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# Gauge invariance, infrared/collinear singularities and tree level matrix element for $e^+e^- \to \nu_e\bar{\nu}_e\gamma\gamma$

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**Abstract.** One of the necessary steps in constructing a high-precision option of KKMC, a Monte Carlo program for the high-precision simulation of fermion pair production at LEP and Linear Collider energies, was to make a careful study of the appropriate matrix elements calculated from QED and the complete standard model. In particular, the installation of the double bremsstrahlung matrix element for the process $e^+e^- \to \nu_e\bar{\nu}_e$ into the scheme of coherent exclusive exponentiation (CEEX) was necessary. In the CEEX scheme one has to define an extrapolation and/or reduction procedure to enable the use of the matrix elements for kinematical configurations with a large number of outgoing particles. The process under study is particularly interesting because of the gauge cancellation of contributions for photon emission from incoming fermion lines and $t$-channel $W$. The QED U(1) gauge properties require terms of the triple and quartic gauge couplings to be taken into consideration as well. A natural separation of the complete amplitude into gauge invariant parts was found and is among the main results of the paper. Each part has a well defined physical interpretation, which after partial integration over phase space provides infrared singular, leading-log, next-to-leading-log, and other terms. Contributions related to the triple and quartic gauge coupling of $W$ (extracted with the help of an expansion around the contact $W$-interaction) have been ordered as well. The separation can be of broader interest; it originates from the rigorous calculation of matrix elements; it visualizes, in the language of spin amplitudes, the properties of factorization necessary for the common multi-process picture. For example, the multiple photon algorithm of PHOTOS, based on the parton shower-like approach, profits from similar considerations. These somewhat speculative aspects of the calculation will be mentioned in the paper as well.

## 1 Introduction

Higher-order radiative corrections are usually necessary to obtain high-precision results for phenomenologically important quantities from the standard model. The techniques of direct calculations lead to expressions of hundreds, thousands, or even millions of terms. These expressions are difficult to control analytically and/or numerically. This is worrisome, because to obtain phenomenologically sound results third-order effects are mandatory; see e.g. [1]. This is clearly outside the reach of presently available methods of direct perturbative calculations. There is no doubt that resummation of at least some contributions from orders higher than the second is necessary.

In the case of electroweak processes at LEP, techniques based on the exclusive exponentiation of QED effects turned out to be powerful and enabled high-precision predictions for a wide range of processes, such as Bhabha scattering, the production of heavy bosons, $W$ or $Z$, and lepton pairs. The underlying method, originating from the pioneering work of Yennie, Frautschi and Suura [2], turned out to be realizable in practice [3–7] thanks to accumulated experience and ever-increasing computer power.

One of the necessary elements in the approach based on exponentiation is a rigorous study of the matrix elements obtained from perturbative calculation. In fact it is not enough to calculate predictions at the highest possible order of the perturbation expansion, but it is also necessary to carefully separate results into infrared singular and remaining finite parts. Thanks to the properties of QED, each order singular and leading of terms can be obtained without explicit perturbative calculations. These leading and universal parts of the amplitudes can also be combined with the phase space into the module of the low level Monte Carlo generator (or in general, into a multidimensional distribution), which can be understood as the lowest order of an improved perturbative expansion. Later, finite parts of the matrix elements can be added order-by-order. In the case of Monte Carlo algorithms, this can be done with the help of a correcting weight, which can be shown to be positive and bounded from above. Details of

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such a scheme can be found in [7, 8]. It improves significantly the convergence of the perturbative expansion: final states with an arbitrary number of photons are present already at the lowest level of the expansion. This allows for predictions for realistic experimental cut-offs. The solution based on separation at the amplitude level is specially useful, because the implementation of interference is easy and convergence of the expansion is particularly fast. This underlying exponentiation scheme is called coherent exclusive exponentiation (CEEX).

In the case of exponentiation, configurations with multiple real photons are present and it may happen that for a particular event there are more explicit photons in the final state than in the expression available from the perturbative expansion. Reduction/extrapolation methods are thus necessary. We will not elaborate on theoretical aspects of this point here; however, let us stress that if a sufficiently high order of the perturbation expansion is available, the dependence on the choice of the reduction procedure or extrapolation drops out. Particularly bad choices may, nonetheless, degrade the convergence of the expansion. Thus, it is important to provide results of perturbative calculations in a form as convenient as possible for the extrapolation procedure. Comparisons of amplitudes calculated at different orders of the perturbative expansion can also provide a useful hint.

One of the principal purposes of the present paper is to provide a missing part of the calculations [7, 8] embodied in the KKMC Monte Carlo for the fermion pair production widely used in the interpretation of the LEP data. Some important theoretical aspects of KKMC specific to the process $e^+e^- \to \nu_e\bar{\nu}_e n(\gamma)$ were not addressed in [7,8], and were covered in [9] at first order only. That is why in the main part of Sect. 2 of the present paper we rely heavily on conventions introduced in [10], and we will assume that the reader has a certain level of familiarity with that reference. The main part of Sect. 2 (starting from (7)) is not essential for a first reading of our paper, if one is interested in the general idea only. Section 2.1 provides the most essential point, also in less hermetic language. On the contrary, Sect. 4 is oriented mainly to the documentation of KKMC and may be skipped at a first reading.

We believe that the results presented in our paper may be of some interest for a wider audience as well. This is why in the remaining sections of the paper we tend to use a more universal notation. We simply use the language of spinors and four-vectors without any specific choices made. Another reason why the internal structure of amplitudes for the processes $e^+e^- \to \nu_e\bar{\nu}_e\gamma\gamma$ and $e^+e^- \to \nu_e\bar{\nu}_e\gamma$ may be of more general interest is that it is particularly rich. Even though all collinear and infrared singular terms have the structure of pure initial-state radiation, emission from the $t$-channel $W$ contributes as well. Thus the pattern of gauge cancellations is complex; triple and quartic gauge couplings contribute.

However, let us note that the spin amplitudes for these processes are well defined within the standard model and have been known for a long time; see e.g. [11]. We could profit in our work from the ready-to-use computer codes, such as [12], available for numerical cross checks of our results.

Our paper is organized as follows. Section 2 is devoted to the case of single bremsstrahlung, $e^+e^- \to \nu_e\bar{\nu}_e\gamma$; some basic elements of spin-amplitude techniques useful in the more complex case of double bremsstrahlung are presented there as well. In Sect. 3, we provide the main results; in particular, we explicitly identify the gauge invariant parts of the amplitudes. We stress points which are useful for the extrapolation schemes used in the CEEX exponentiation as well. We keep our discussion in as universal a language as possible, having in mind future applications in automated spin-amplitude programs. In Sect. 4 we discuss issues related to the extrapolation procedure in more detail. Finally, Sect. 5 summarizes the paper.

## 2 Amplitude for one real photon and notation

Let us start with the well-known and straightforward calculation of $\mathcal{O}(\alpha)$ spin amplitude for the $e^+e^- \to \nu_e\bar{\nu}_e\gamma$ single-photon bremsstrahlung process; see Fig. 1. We will recall it here, in order to define the framework for our discussion. The conventions of [8–10] are used. Let us recall here only the most important notation. The four-momenta $p_a$, $p_b$, $p_c$, $p_d$, $k_1$ denote respectively the momenta of incoming electron, positron, outgoing neutrino, antineutrino and photon. The indices for the spin states for the fermions are denoted respectively $\lambda_a$, $\lambda_b$, $\lambda_c$, $\lambda_d$ and for the photon $\sigma_1$. The photon polarization vector is denoted $\epsilon_{\sigma_1}$. The gauge transformation in our case reduces to the replacement $\epsilon_{\sigma_1} \to \epsilon_{\sigma_1} + x\, k_1$ (with arbitrary coefficient $x$), there will be no external boson lines, and incoming fermions lead to trivial phases only. With these notations the first-order matrix element,[1] obtained from the Feynman diagrams

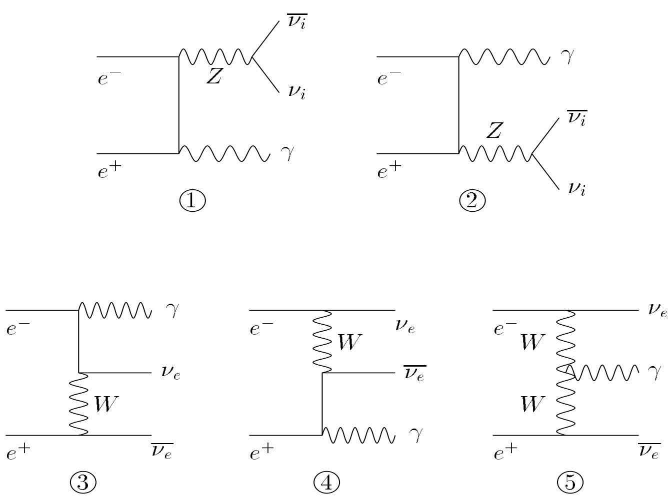


**Fig. 1.** The Feynman diagrams for $e^+e^- \to \bar{\nu}_e\nu_e\ \gamma$

[1] $\mathcal{M}_{1\{\mathrm{I}\}}\binom{p\,k_1}{\lambda\,\sigma_1}$: the subscripts 1 and {I} denote, respectively, that the amplitudes are of the first order and are included as part of the initial-state bremsstrahlung. This spurious notation is convenient for the reader interested in [10].

depicted in Fig. 1, can be written in a rather straightforward way:

$$
\begin{aligned}
&\mathcal{M}_{1\{\mathrm{I}\}}\left(\begin{smallmatrix} p\, k_1 \\ \lambda\, \sigma_1\end{smallmatrix}\right)\\
&= eQ_e \bar{v}(p_b,\lambda_b)\mathbf{M}^{bd}_{\{\mathrm{I}\}} \frac{\not p_a + m - \not k_1}{-2k_1p_a}\ \not\epsilon^\star_{\sigma_1}(k_1)u(p_a,\lambda_a)\\
&\quad + eQ_e\bar{v}(p_b,\lambda_b)\ \not\epsilon^\star_{\sigma_1}(k_1)\frac{-\not p_b + m + \not k_1}{-2k_1p_b}\mathbf{M}^{ac}_{\{\mathrm{I}\}}u(p_a,\lambda_a)\\
&\quad + e\bar{v}(p_b,\lambda_b)\mathbf{M}^{bd,ac}_{\{\mathrm{I}\}}u(p_a,\lambda_a)\\
&\quad \times \frac{\epsilon^\star_{\sigma_1}(k_1)\cdot(p_c - p_a + p_b - p_d)}{(t_a - M_W^2)(t_b - M_W^2)}\\
&\quad + e\frac{\bar{v}(p_b,\lambda_b)g^{We\nu}_{\lambda_b,\lambda_d}\ \not\epsilon^\star_{\sigma_1}(k_1)v(p_d,\lambda_d)}{(t_a - M_W^2)(t_b - M_W^2)}\\
&\quad \times \bar{u}(p_c,\lambda_c)g^{We\nu}_{\lambda_c,\lambda_a}\ \not k_1 u(p_a,\lambda_a)\\
&\quad - e\frac{\bar{v}(p_b,\lambda_b)g^{We\nu}_{\lambda_b,\lambda_d}\ \not k_1 v(p_d,\lambda_d)}{(t_a - M_W^2)(t_b - M_W^2)}\\
&\quad \times \bar{u}(p_c,\lambda_c)g^{We\nu}_{\lambda_c,\lambda_a}\ \not\epsilon^\star_{\sigma_1}(k_1)u(p_a,\lambda_a),
\end{aligned}
\tag{1}
$$

or, equivalently,

$$
\begin{aligned}
&\mathcal{M}_{1\{\mathrm{I}\}}\left(\begin{smallmatrix} p\, k_1 \\ \lambda\, \sigma_1\end{smallmatrix}\right) = \mathcal{M}^0 + \mathcal{M}^1 + \mathcal{M}^2 + \mathcal{M}^3,\\
&\mathcal{M}^0 = eQ_e\\
&\quad \times \bar{v}(p_b,\lambda_b)\mathbf{M}^{bd}_{\{\mathrm{I}\}} \frac{\not p_a + m - \not k_1}{-2k_1p_a}\ \not\epsilon^\star_{\sigma_1}(k_1)u(p_a,\lambda_a)\\
&\quad + eQ_e\bar{v}(p_b,\lambda_b)\ \not\epsilon^\star_{\sigma_1}(k_1)\frac{-\not p_b + m + \not k_1}{-2k_1p_b}\mathbf{M}^{ac}_{\{\mathrm{I}\}}u(p_a,\lambda_a),\\
&\mathcal{M}^1 = \mathcal{M}^{1'} + \mathcal{M}^{1''},\\
&\mathcal{M}^{1'} = +e\bar{v}(p_b,\lambda_b)\mathbf{M}^{bd,ac}_{\{\mathrm{I}\}}u(p_a,\lambda_a)\epsilon^\star_{\sigma_1}(k_1)\cdot(p_c - p_a)\\
&\quad \times \frac{1}{t_a - M_W^2}\,\frac{1}{t_b - M_W^2},\\
&\mathcal{M}^{1''} = +e\bar{v}(p_b,\lambda_b)\mathbf{M}^{bd,ac}_{\{\mathrm{I}\}}u(p_a,\lambda_a)\epsilon^\star_{\sigma_1}(k_1)\cdot(p_b - p_d)\\
&\quad \times \frac{1}{t_a - M_W^2}\,\frac{1}{t_b - M_W^2},\\
&\mathcal{M}^2 = +e\bar{v}(p_b,\lambda_b)g^{We\nu}_{\lambda_b,\lambda_d}\ \not\epsilon^\star_{\sigma_1}(k_1)v(p_d,\lambda_d)\\
&\quad \times \bar{u}(p_c,\lambda_c)g^{We\nu}_{\lambda_c,\lambda_a}\ \not k_1 u(p_a,\lambda_a)\frac{1}{t_a - M_W^2}\,\frac{1}{t_b - M_W^2},\\
&\mathcal{M}^3 = -e\bar{v}(p_b,\lambda_b)g^{We\nu}_{\lambda_b,\lambda_d}\ \not k_1 v(p_d,\lambda_d)\\
&\quad \times \bar{u}(p_c,\lambda_c)g^{We\nu}_{\lambda_c,\lambda_a}\ \not\epsilon^\star_{\sigma_1}(k_1)u(p_a,\lambda_a)\frac{1}{t_a - M_W^2}\,\frac{1}{t_b - M_W^2}.
\end{aligned}
\tag{2}
$$

Part of the amplitude consisting of bosonic couplings (the $g^{Z,f}_\lambda$ denote the coupling constant of $Z$ to fermion $f$ of handedness $\lambda$, in units of electric charge), spinors for final-state fermions and boson propagators reads

$$
\mathbf{M}^{xy}_{\{\mathrm{I}\}} = \mathrm{i}e^2(\mathcal{R}_Z + \mathcal{R}_W) = \mathrm{i}e^2 \sum_{B=W,Z} \varPi^{\mu\nu}_B(X)G^B_{e,\mu}(G^B_{f,\nu})_{[cd]}
\tag{3}
$$

with

$$
\begin{aligned}
&G^B_{e,\mu} = \gamma_\mu \sum_{\lambda=\pm}\frac{1}{2}(1+\lambda\gamma_5)g^{B,e}_\lambda,\\
&(G^B_{f,\nu})_{[cd]} = \bar{u}(p_c,\lambda_c)G^B_{f,\nu}v(p_d,\lambda_d),\\
&\varPi^{\mu\nu}_{B=Z}(X) = \frac{g^{\mu\nu}}{X^2 - M_Z^2 + \mathrm{i}\Gamma_Z X^2/M_Z},\\
&\varPi^{\mu\nu}_{B=W}(X) = \frac{g^{\mu\nu}}{t - M_W^2}.
\end{aligned}
\tag{4}
$$

The final-state spinors are explicitly included, and a Fierz transformation is applied for the part of $W$ exchange. The $W$ coupling constant reads

$$
g^{We\nu}_{\lambda_c,\lambda_a} = \frac{1}{\sqrt{2}\sin\theta_\mathrm{W}}\delta^{\lambda_c}_{\lambda_a}\delta^{\lambda_c}_{+}.
\tag{5}
$$

Only for the $W$ contribution, the superscripts $xy$ in $\mathbf{M}_{\{\mathrm{I}\}}$ have a meaning; they define the momentum transfer in the $W$ propagator $\varPi^{\mu\nu}_W(X)$: for $xy = ac$ the transfer[2] is $t_a = (p_a - p_c)^2$, and for $bd$ it is $t_b = (p_b - p_d)^2$. If both are explicitly marked, then the expression

$$
\mathbf{M}^{bd,ac}_{\{\mathrm{I}\}} = \mathrm{i}e^2 G^W_{e,\mu}(G^{W,\mu}_\nu)_{[cd]}
\tag{6}
$$

is used. For those parts of formula (2) the $W$ propagators are explicitly given. The symbols $\mathcal{R}_Z$ and $\mathcal{R}_W$ will be defined later; see (25) and (28).

Let us rewrite expression (2). It is straightforward to notice that the first term $\mathcal{M}^0$ can be split into soft IR parts proportional to $(\not p \pm m)$ and non-IR parts proportional to $\not k_1$. The non-IR parts are individually gauge invariant by construction. The soft part of $\mathcal{M}^0$, with $Z$ couplings only, is gauge invariant as well.

Employing the completeness relations of (A14) from [8] we obtain the following form[3] of (2):

$$
\begin{aligned}
&\mathcal{M}_{1\{\mathrm{I}\}}\left(\begin{smallmatrix} p\, k_1 \\ \lambda\, \sigma_1\end{smallmatrix}\right)\\
&= -\frac{eQ_e}{2k_1p_a}\sum_{\rho_a}\mathfrak{B}\left[\begin{smallmatrix} p_b\, p_a \\ \lambda_b\, \rho_a\end{smallmatrix}\right]_{[cd]} U\left[\begin{smallmatrix} p_a\, k_1\, p_a \\ \rho_a\, \sigma_1\, \lambda_a\end{smallmatrix}\right]\\
&\quad + \frac{eQ_e}{2k_1p_b}\sum_{\rho_b} V\left[\begin{smallmatrix} p_b\, k_1\, p_b \\ \lambda_b\, \sigma_1\, \rho_b\end{smallmatrix}\right]\mathfrak{B}\left[\begin{smallmatrix} p_b\, p_a \\ \rho_b\, \lambda_a\end{smallmatrix}\right]_{[cd]}\\
&\quad + \frac{eQ_e}{2k_1p_a}\sum_{\rho}\mathfrak{B}\left[\begin{smallmatrix} p_b\, k_1 \\ \lambda_b\, \rho\end{smallmatrix}\right]_{[cd]} U\left[\begin{smallmatrix} k_1\, k_1\, p_a \\ \rho\, \sigma_1\, \lambda_a\end{smallmatrix}\right]\\
&\quad - \frac{eQ_e}{2k_1p_b}\sum_{\rho} V\left[\begin{smallmatrix} p_b\, k_1\, k_1 \\ \lambda_b\, \sigma_1\, \rho\end{smallmatrix}\right]\mathfrak{B}\left[\begin{smallmatrix} k_1\, p_a \\ \rho\, \lambda_a\end{smallmatrix}\right]_{[cd]}
\end{aligned}
$$

[2] Transfers can be expressed also as $t_a = (p_b - k_1 - p_d)^2$ and $t_b = (p_a - k_1 - p_c)^2$, this make difference if extrapolation procedures are used for the configurations off mass shell where $p_a + p_b \neq p_c + p_d + k_1$, otherwise $\mathcal{M}^{1'} = \mathcal{M}^{1''}$ of course.

[3] Note that the differences in fonts: $\mathcal{M}$, $\mathcal{M}$ and $\mathbf{M}$ are significant, the symbols corresponding respectively to complete spin amplitude, additive part of the amplitude and finally the part describing the hard interaction alone.

$$+\mathcal{M}^{1'} + \mathcal{M}^{1''} + \mathcal{M}^2 + \mathcal{M}^3. \tag{7}$$

The terms $\mathcal{M}^{1'}$ to $\mathcal{M}^3$ correspond to the last three lines[4] of (1). These contributions are infrared-finite (IR-finite).

In the next step let us remove the sum in the first two terms thanks to the diagonality of $U$ and $V$ [8]. The matrices $\mathfrak{B}$ are also defined in this reference. We obtain

$$\begin{aligned}
\mathfrak{M}_{1\{\mathrm{I}\}}\left(\begin{smallmatrix} p\, k_1 \\ \lambda\, \sigma_1 \end{smallmatrix}\right) &= \mathfrak{s}^{\{\mathrm{I}\}}_{\sigma_1}(k_1)\hat{\mathfrak{B}}\left[\begin{smallmatrix} p \\ \lambda \end{smallmatrix}\right] + \left(r^{B'}_{\{\mathrm{I}\}} + \mathcal{M}^{1'}\right) \\
&\quad + \left(r^{B''}_{\{\mathrm{I}\}} + \mathcal{M}^{1''}\right) + r^{A'}_{\{\mathrm{I}\}} + r^{A''}_{\{\mathrm{I}\}} \\
&\quad + \left(\mathcal{M}^2 + \mathcal{M}^3\right), \qquad (8) \\
r^{B'}_{\{\mathrm{I}\}}\left(\begin{smallmatrix} p\, k_1 \\ \lambda\, \sigma_1 \end{smallmatrix}\right) &= -\frac{eQ_e}{2k_1p_a}\sum_\rho \bar{\mathfrak{B}}\left[\begin{smallmatrix} p_b\, p_a \\ \lambda_b\, \rho_a \end{smallmatrix}\right]_{[cd]} U\left[\begin{smallmatrix} p_a\, k_1\, p_a \\ \rho_a\, \sigma_1\, \lambda_a \end{smallmatrix}\right], \\
r^{B''}_{\{\mathrm{I}\}}\left(\begin{smallmatrix} p\, k_1 \\ \lambda\, \sigma_1 \end{smallmatrix}\right) &= +\frac{eQ_e}{2k_1p_b}\sum_\rho V\left[\begin{smallmatrix} p_b\, k_1\, p_b \\ \lambda_b\, \sigma_1\, \rho_b \end{smallmatrix}\right] \bar{\mathfrak{B}}\left[\begin{smallmatrix} p_b\, p_a \\ \rho_b\, \lambda_a \end{smallmatrix}\right]_{[cd]}, \\
r^{A'}_{\{\mathrm{I}\}}\left(\begin{smallmatrix} p\, k_1 \\ \lambda\, \sigma_1 \end{smallmatrix}\right) &= +\frac{eQ_e}{2k_1p_a}\sum_\rho \mathfrak{B}\left[\begin{smallmatrix} p_b\, k_1 \\ \lambda_b\, \rho \end{smallmatrix}\right]_{[cd]} U\left[\begin{smallmatrix} k_1\, k_1\, p_a \\ \rho\, \sigma_1\, \lambda_a \end{smallmatrix}\right], \\
r^{A''}_{\{\mathrm{I}\}}\left(\begin{smallmatrix} p\, k_1 \\ \lambda\, \sigma_1 \end{smallmatrix}\right) &= -\frac{eQ_e}{2k_1p_b}\sum_\rho V\left[\begin{smallmatrix} p_b\, k_1\, k_1 \\ \lambda_b\, \sigma_1\, \rho \end{smallmatrix}\right] \mathfrak{B}\left[\begin{smallmatrix} k_1\, p_a \\ \rho\, \lambda_a \end{smallmatrix}\right]_{[cd]}, \\
\mathfrak{s}^{\{\mathrm{I}\}}_{\sigma_1}(k_1) &= -eQ_e\frac{b_{\sigma_1}(k_1,p_a)}{2k_1p_a} + eQ_e\frac{b_{\sigma_1}(k_1,p_b)}{2k_1p_b}. \qquad (9)
\end{aligned}$$

The soft part is now clearly separated from the remaining non-IR part, used in the CEEX exponentiation for the construction of $\mathcal{O}(\alpha)$ corrections. We have ordered the expression, with the help of an expansion similar to the contact interaction for the $W$ propagator as well. In $\hat{\mathfrak{B}}\left[\begin{smallmatrix} p \\ \lambda \end{smallmatrix}\right]$ we use an auxiliary fixed transfer $t_0$, which is independent of the place where the photon is attached to the fermion line. In fact $t_0$ is arbitrary and the choice $t_0 = 0$ could be used as well[5]. With the help of $\bar{\mathfrak{B}}$ we provide the residual contribution calculated as the difference of the expression calculated with the true $t$-transfers ($t_a$ or $t_b$) and the auxiliary $t_0$ one. Note that $\mathfrak{B} = \hat{\mathfrak{B}} + \bar{\mathfrak{B}}$. Each of the contributions to the sum given in the first equation of (8) is independently gauge invariant.

We can see [9] that it was possible to separate the complete spin amplitude for the process $e^+e^- \to \bar{\nu}_e\nu_e\gamma$ into *six* individually QED gauge invariant parts. This conclusion is rather straightforward to check, replacing the photon polarization vector with its four-momentum, in every element of the master sum in (8). Also, each element has a rather well defined physical interpretation. It is also easy to verify that the gauge invariance of each part can be preserved in the case of the extrapolation, when the condition $p_a + p_b = p_c + p_d + k_1$ is not valid. Let us elaborate on this point a bit more.

(1) The first term of the soft photon type $\mathfrak{s}^{\{\mathrm{I}\}}_{\sigma_1}(k_1)\hat{\mathfrak{B}}\left[\begin{smallmatrix} p \\ \lambda \end{smallmatrix}\right]$ is gauge invariant thanks to the invariance of the standard ISR soft factor $\mathfrak{s}^{\{\mathrm{I}\}}_{\sigma_1}(k_1)$. It is also of the universal form, identical for the diagrams with the $s$-channel $Z$ exchange as well as the $t$-channel $W$.
(2) The next two terms $\left(r^{B'}_{\{\mathrm{I}\}} + \mathcal{M}^{1'}\right)$ and $\left(r^{B''}_{\{\mathrm{I}\}} + \mathcal{M}^{1''}\right)$ originate only from diagrams of $t$-channel $W$ exchange. For the gauge invariance to hold, the $t$-channel transfers have to be $t_a = (p_a - p_c)^2$, $t_b = (p_a - k_1 - p_c)^2$ for the first term (and $t_a = (p_b - k_1 - p_d)^2$, $t_b = (p_b - p_d)^2$ for the second one)[6].
(3) The consecutive two terms $r^{A'}_{\{\mathrm{I}\}}\left(\begin{smallmatrix} p\, k_1 \\ \lambda\, \sigma_1 \end{smallmatrix}\right)$ and $r^{A''}_{\{\mathrm{I}\}}\left(\begin{smallmatrix} p\, k_1 \\ \lambda\, \sigma_1 \end{smallmatrix}\right)$ are again of the same universal form as for any $s$-channel process and are gauge invariant by construction. These are also the terms which lead to leading-log (but not infrared) singular terms after phase space integration. There the photon polarization vector and its momentum stand side-by-side.
(4) Finally, for the last expression $\left(\mathcal{M}^2 + \mathcal{M}^3\right)$ to be gauge invariant it is enough that in both terms the choices for $t_a$, $t_b$ are identical; the same reduction procedure[7] is used.
(5) Note that the relation of the Born level amplitude in the contact approximation for $W$ exchange and the complete amplitude for single-photon emission is clearly visible and enables one to make a physical interpretation of the expression obtained.

### 2.1 Simplest case of $e^+e^- \to \nu_\mu\bar{\nu}_\mu$

Let us finish this section with a discussion of the $Z$ exchange part of the amplitude (2) in the simple language of spinors and four-vectors. This part of the amplitude (and the language) is important because it will define the framework for our main results collected in Sect. 3. We have

$$\begin{aligned}
\mathfrak{M}^Z_{1\{\mathrm{I}\}}\left(\begin{smallmatrix} p\, k_1 \\ \lambda\, \sigma_1 \end{smallmatrix}\right) &= eQ_e \qquad (10) \\
&\times\bar{v}(p_b,\lambda_b)\mathbf{M}_{\{\mathrm{I}\}}\frac{\not{p}_a + m - \not{k}_1}{-2k_1p_a}\, \not{\epsilon}^\star_{\sigma_1}(k_1)u(p_a,\lambda_a) \\
&+eQ_e\bar{v}(p_b,\lambda_b)\, \not{\epsilon}^\star_{\sigma_1}(k_1)\frac{-\not{p}_b + m + \not{k}_1}{-2k_1p_b}\mathbf{M}_{\{\mathrm{I}\}}u(p_a,\lambda_a);
\end{aligned}$$

the superscript $ac$ or $bd$ in $\mathbf{M}_{\{\mathrm{I}\}}$ can be dropped, because here $\mathbf{M}_{\{\mathrm{I}\}}$ does not depend on the $t$-dependent $W$ propagator. For the precise specification of the part of amplitude for the hard interaction $\mathbf{M}_{\{\mathrm{I}\}} = \mathcal{R}_Z$, see (25), later in the text.

[4] The term $\mathcal{M}^1 + \mathcal{M}^2 + \mathcal{M}^3$ originates from the $WW\gamma$ vertex

$$-\mathrm{i}e\left[g_{\mu\nu}(p-q)_\rho + g_{\nu\rho}(q-r)_\mu + g_{\mu\rho}(r-p)_\nu\right]$$

where all momenta are outgoing, and indices on outgoing lines are paired with the momenta $p^\mu$, $q^\nu$ $r^\rho$; $\mathcal{M}^1$ originates from the term where $g^{\mu\nu}$ connects the $e^-$–$\nu_e$, $e^+$–$\bar{\nu}_e$ fermion lines.

[5] The choice is nonetheless important from the point of view of efficiency; it affects the size of the corrections in the CEEX expansion. The condition that $\lim_{k_1\to 0} t_{a/b} = t_0$ is desirable.

[6] It is interesting to realize that only part of the diagram is involved in the cancellation; for example emission from the upper fermion and boson lines only. This observation will become useful in case of double bremsstrahlung amplitudes.

[7] Mechanism of gauge cancellation is fulfilled already at the level of bosonic interaction alone. Also this observation will be useful in a study of double bremsstrahlung amplitudes.

The gauge invariance of the two sub-parts proportional to $\not{k}_1$ is straightforward to see, because the product $\not{k}_1\ \not{\epsilon}^\star_{\sigma_1}(k_1)$ alone, is gauge invariant. These parts of the amplitude do not contribute to the infrared singularity; however, they do contribute to the large logarithm related to the collinear singularity (once amplitudes are squared and integrated over the phase space). That is why we will refer to these parts of the amplitude as infrared-finite collinear singular. The remaining part of the amplitude,

$$\mathcal{M}^{Z-ir}_{1\{\mathrm{I}\}}\binom{p\,k_1}{\lambda\,\sigma_1} \tag{11}$$
$$= eQ_e\bar{v}(p_b,\lambda_b)\mathbf{M}^{bd}_{\{\mathrm{I}\}}\frac{\not{p}_a+m}{-2k_1p_a}\ \not{\epsilon}^\star_{\sigma_1}(k_1)u(p_a,\lambda_a)$$
$$+eQ_e\bar{v}(p_b,\lambda_b)\ \not{\epsilon}^\star_{\sigma_1}(k_1)\frac{-\not{p}_b+m}{-2k_1p_b}\mathbf{M}^{ac}_{\{\mathrm{I}\}}u(p_a,\lambda_a),$$

factorizes, thanks to the orthogonality for Dirac spinors,

$$\not{p}_a+m=\sum_\lambda u(p_a,\lambda)\bar{u}(p_a,\lambda)$$
$$-\not{p}_b+m=-\sum_\lambda v(p_b,\lambda)\bar{v}(p_b,\lambda), \tag{12}$$

into a gauge invariant soft photon factor and a Born amplitude:

$$\mathcal{M}^{Z-ir}_{1\{\mathrm{I}\}}\binom{p\,k_1}{\lambda\,\sigma_1}=\mathfrak{s}^{\{\mathrm{I}\}}_{\sigma_1}(k_1)\bar{v}(p_b,\lambda_b)\mathbf{M}^{bd}_{\{\mathrm{I}\}}u(p_a,\lambda_a),$$
$$\mathfrak{s}^{\{\mathrm{I}\}}_{\sigma_1}(k_1)=+\frac{eQ_e}{-2k_1p_a}\bar{u}(p_a,\lambda)\ \not{\epsilon}^\star_{\sigma_1}(k_1)u(p_a,\lambda_a)$$
$$+\frac{eQ_e}{2k_1p_b}\bar{v}(p_b,\lambda_b)\ \not{\epsilon}^\star_{\sigma_1}(k_1)v(p_b,\lambda) \tag{13}$$
$$=\frac{-eQ_e}{2k_1p_a}b_{\sigma_1}(k_1,p_a)\delta_{\lambda\lambda_a}+\frac{eQ_e}{2k_1p_b}b_{\sigma_1}(k_1,p_b)\delta_{\lambda\lambda_b}.$$

The gauge invariance takes place in case of $Z$ exchange (and also $W$ exchange, if the approximation of a contact interaction is used) because $\mathbf{M}_{\{\mathrm{I}\}}=\mathbf{M}^{bd}_{\{\mathrm{I}\}}=\mathbf{M}^{ac}_{\{\mathrm{I}\}}$. Also, the two parts of $\mathfrak{s}^{\{\mathrm{I}\}}_{\sigma_1}(k_1)$ are diagonal respectively in a pair of indices $\lambda\lambda_{a(b)}$. The Born level spin amplitude factorizes out, and the gauge dependent soft factors for the emission from electron and positron lines can be summed to a gauge invariant $\mathfrak{s}^{\{\mathrm{I}\}}_{\sigma_1}(k_1)$. For the explicit definition of $b_{\sigma_1}(k_1,p_b)$, see e.g. formula (231) of [8]. We will use the factorization of the soft factors, explained here, later in the paper as well.

Note that in the case of the single-photon $Z$ exchange amplitude, we have only three gauge invariant parts: an infrared-singular one and two other ones contributing to collinear-singular terms (after phase space integration). The residual terms (contributing only non-enhanced terms after phase space integration) are absent. Such terms are present in case of the $W$ exchange.

Finally, let us comment that a similar pattern of amplitude separation into gauge invariant parts can be observed for $W^\pm \to l\nu_l\gamma$ [13].

## 3 Double bremsstrahlung

In the present section we will study the amplitudes for the double bremsstrahlung in $e^+e^- \to \nu_e\bar{\nu}_e$ production process. There are two individually gauge invariant groups of diagrams in this case. The first has $Z$ boson exchange in the $s$-channel, and the second one has $t$-channel $W$ exchange. Similarly, as in previous section and single bremsstrahlung, we will check if gauge invariant parts of the complete amplitude can be defined. Also we will be interested if this can be done in a semi-automatic way, directly from the Feynman rules.

The presentation of the complete amplitudes is cumbersome because of their length. To avoid lengthy formulae, we will start with a largely incomplete set of diagrams, which is nonetheless sufficient to localize some gauge invariant group of terms. Once localized, it will be hidden under the symbol (or group of symbols) $L^b_a$ and left behind. To the remaining gauge dependent part, the contributions from the next diagrams will be added. Again, a gauge invariant group of terms will be sought. This procedure will be repeated until the complete list of diagrams of our process is exhausted. The choice for the first diagram in this procedure is motivated by its particular (unique) form. Later steps are motivated by the form of the gauge dependent remainder from the previous one[8]. For short hand notations we will use extended subscripts and superscripts for $L^b_a$. For example, we will use the symbol $L^{k_1,k_2}_{e^-}(n)$, to denote the contribution for the diagram with the first photon of momentum, $k_1$, and the second one, $k_2$, attached to the incoming $e^-$ line. The number $n$ in brackets (if present) will denote that it may be only a part of the contribution from the particular Feynman diagram (or diagrams). A bar over this number $n$ means that the particular part is gauge dependent.

Let us start our iteration with diagrams involving the double fermion propagator, that is, diagrams where two photons are attached either to an incoming electron or to an incoming positron. These are the *only* diagrams with a $k_1\cdot k_2$ term in the fermion propagators. Our first aim will be to localize the parts which are gauge invariant by themselves and include this $k_1\cdot k_2$ term. Let us consider the eight diagrams with the photon lines attached either to electron or positron line; see Fig. 2. Explicitly, we will write down the part of the amplitude corresponding to the incoming electron line only. The diagrams with $Z$ and $W$ exchange are quite similar:

$$L^{k_1,k_2}_{e^-}=(eQ_e)^2$$
$$\times\bar{v}(p_b,\lambda_b)\mathcal{R}_B\left(\frac{\not{p}_a+m-\not{k}_1-\not{k}_2}{-2k_1p_a-2k_2p_a-2k_1k_2}\ \not{\epsilon}^\star_{\sigma_2}\right.$$
$$\times\frac{\not{p}_a+m-\not{k}_1}{-2k_1p_a}\ \not{\epsilon}^\star_{\sigma_1}u(p_a,\lambda_a)$$
$$+\frac{\not{p}_a+m-\not{k}_1-\not{k}_2}{-2k_1p_a-2k_2p_a-2k_1k_2}\ \not{\epsilon}^\star_{\sigma_1}$$

[8] We suspect that the method presented here can be automated and applied to other processes and at higher orders of the perturbation expansion as well.

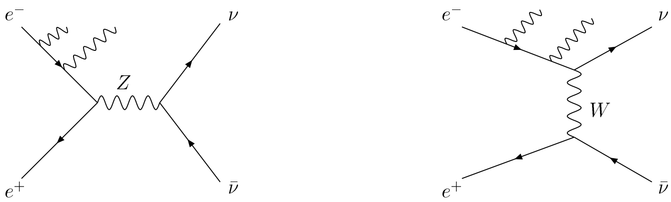


**Fig. 2.** Double emission from electron

$$\times \frac{\not{p}_a + m - \not{k}_2}{-2k_2p_a} \not{\epsilon}^\star_{\sigma_2} u(p_a,\lambda_a)\Bigg) . \tag{14}$$

Only the expression $\mathcal{R}_B$, describing a final-state neutrino interaction either with $Z$ or $W$ distinguishes the two cases; it is defined later, in (25) and (28) respectively. We can separate (14) into the following parts:

$$L_{e^-}^{k_1,k_2} = L_{e^-}^{k_1,k_2}(1) + L_{e^-}^{k_1,k_2}(2) + L_{e^-}^{k_1,k_2}(\bar{3}) + L_{e^-}^{k_1,k_2}(\bar{4}), \tag{15}$$

where

$$\begin{aligned} L_{e^-}^{k_1,k_2}(1) &= (eQ_e)^2 \\ &\times \bar{v}(p_b,\lambda_b)\mathcal{R}_B \Bigg( \frac{-\not{k}_2}{-2k_1p_a - 2k_2p_a - 2k_1k_2} \not{\epsilon}^\star_{\sigma_2}(k_2) \\ &\times \frac{-\not{k}_1}{-2k_1p_a} \not{\epsilon}^\star_{\sigma_1}(k_1) u(p_a,\lambda_a) \\ &+ \frac{-\not{k}_1}{-2k_1p_a - 2k_2p_a - 2k_1k_2} \not{\epsilon}^\star_{\sigma_1}(k_1) \\ &\times \frac{-\not{k}_2}{-2k_2p_a} \not{\epsilon}^\star_{\sigma_2}(k_2) u(p_a,\lambda_a) \Bigg) \end{aligned} \tag{16}$$

is gauge invariant by construction, thanks to the terms $\not{k}_1\ \not{\epsilon}^\star_{\sigma_1}(k_1)$ and $\not{k}_2\ \not{\epsilon}^\star_{\sigma_2}(k_2)$. This is similar to the case of single bremsstrahlung. The second part,

$$\begin{aligned} L_{e^-}^{k_1,k_2}(2) &= (eQ_e)^2 \\ &\times \bar{v}(p_b,\lambda_b)\mathcal{R}_B \Bigg( \frac{\not{p}_a + m - \not{k}_2}{-2k_1p_a - 2k_2p_a - 2k_1k_2} \not{\epsilon}^\star_{\sigma_1}(k_1) \\ &\times \frac{-\not{k}_2}{-2k_2p_a} \not{\epsilon}^\star_{\sigma_2}(k_2) u(p_a,\lambda_a) \\ &+ \frac{-\not{k}_2}{-2k_1p_a - 2k_2p_a - 2k_1k_2} \not{\epsilon}^\star_{\sigma_1}(k_1) \\ &\times \frac{\not{p}_a + m}{-2k_2p_a} \not{\epsilon}^\star_{\sigma_2}(k_2) u(p_a,\lambda_a) \\ &- \not{k}_2 \left( \frac{1}{-2k_1p_a - 2k_2p_a - 2k_1k_2} - \frac{1}{-2k_2p_a} \right) \\ &\times \not{\epsilon}^\star_{\sigma_2}(k_2) \frac{\not{p}_a + m}{-2k_1p_a} \not{\epsilon}^\star_{\sigma_1}(k_1) u(p_a,\lambda_a) \\ &+ (\not{p}_a + m) \\ &\times \left( \frac{1}{-2k_1p_a - 2k_2p_a - 2k_1k_2} - \frac{1}{-2k_1p_a - 2k_2p_a} \right) \\ &\times \not{\epsilon}^\star_{\sigma_2}(k_2) \frac{\not{p}_a + m}{-2k_1p_a} \not{\epsilon}^\star_{\sigma_1}(k_1) u(p_a,\lambda_a) \Bigg) + (1 \leftrightarrow 2), \end{aligned} \tag{17}$$

is also gauge invariant, but it needs to be checked by direct calculation. This contribution, like the previous one, is free of an infrared singularity. In the definition of $L_{e^-}^{k_1,k_2}(2)$ we had to introduce a subtraction: the terms proportional to $\not{k}_2 \left(-\frac{1}{-2k_2p_a}\right)$ and $(\not{p}_a + m)\left(-\frac{1}{-2k_1p_a - 2k_2p_a}\right)$. The subtracted terms are added back to (15) as (18) and (19), but with the opposite signs of course. It is important to realize, that the form of these subtracted terms is defined uniquely by the $Z$ exchange part of the amplitude for single-photon emission[9] (see (10)) and by the soft photon factor for the second photon.

The term $L_{e^-}^{k_1,k_2}(\bar{3})$ equals

$$\begin{aligned} L_{e^-}^{k_1,k_2}(\bar{3}) &= (eQ_e)^2 \\ &\times \bar{v}(p_b,\lambda_b)\mathcal{R}_B \Bigg( \frac{\not{p}_a + m}{-2k_1p_a - 2k_2p_a} \not{\epsilon}^\star_{\sigma_2}(k_2) \\ &\times \frac{\not{p}_a + m}{-2k_1p_a} \not{\epsilon}^\star_{\sigma_1}(k_1) u(p_a,\lambda_a) \\ &+ \frac{\not{p}_a + m}{-2k_1p_a - 2k_2p_a} \not{\epsilon}^\star_{\sigma_1}(k_1) \frac{\not{p}_a + m}{-2k_2p_a} \not{\epsilon}^\star_{\sigma_2}(k_2) u(p_a,\lambda_a) \Bigg) \\ &= (eQ_e)^2 \bar{v}(p_b,\lambda_b)\mathcal{R}_B \\ &\times \frac{\not{p}_a + m}{-2k_2p_a} \not{\epsilon}^\star_{\sigma_2}(k_2) \frac{\not{p}_a + m}{-2k_1p_a} \not{\epsilon}^\star_{\sigma_1}(k_1) u(p_a,\lambda_a), \end{aligned} \tag{18}$$

and the next term $L_{e^-}^{k_1,k_2}(\bar{4})$ is also free of $k_1k_2$. Its numerator is linear in the photon momentum:

$$\begin{aligned} L_{e^-}^{k_1,k_2}(\bar{4}) &= (eQ_e)^2 \\ &\times \bar{v}(p_b,\lambda_b)\mathcal{R}_B \Bigg( \frac{-\not{k}_2}{-2k_2p_a} \not{\epsilon}^\star_{\sigma_2}(k_2) \\ &\times \frac{\not{p}_a + m}{-2k_1p_a} \not{\epsilon}^\star_{\sigma_1}(k_1) u(p_a,\lambda_a) \\ &+ \frac{-\not{k}_1}{-2k_1p_a} \not{\epsilon}^\star_{\sigma_1}(k_1) \frac{\not{p}_a + m}{-2k_2p_a} \not{\epsilon}^\star_{\sigma_2}(k_2) u(p_a,\lambda_a) \Bigg) . \end{aligned} \tag{19}$$

The complete contribution from the diagrams of Fig. 2, formula (15), is not gauge invariant. The last two terms are gauge dependent, and are also relatively short. The first one, (18), has the structure of a Born amplitude multiplied by soft photon factors. The second one, (19), has the structure of soft photon emission for one of the two photons only; see the discussion at the end of Sect. 2.

Once we have completed the diagrams with two photon lines attached to the same fermion line, let us turn to another group of diagrams, where one of the photons is attached to an electron and another one to a positron line; see Fig. 3. Note that for the subgroup of diagrams with $Z$ boson exchange, these are the last contributing diagrams:

[9] The subtraction term for the $W$ exchange differs only by the replacement $\mathcal{R}_B = \mathcal{R}_W$.

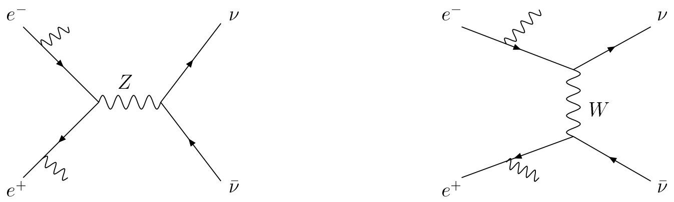


**Fig. 3.** Single emission from electron and positron

$$
\begin{aligned}
L^{k_1,k_2}_{e^-,e^+} &= (eQ_e)^2 \\
&\times \Bigg( \bar{v}(p_b,\lambda_b)\ \not\epsilon^\star_{\sigma_2} \frac{-\not p_b + m + \not k_2}{-2k_2p_b} \mathcal{R}_B \\
&\times \frac{\not p_a + m - \not k_1}{-2k_1p_a}\ \not\epsilon^\star_{\sigma_1} u(p_a,\lambda_a) \\
&+ \bar{v}(p_b,\lambda_b)\ \not\epsilon^\star_{\sigma_1} \frac{-\not p_b + m + \not k_1}{-2k_1p_b} \mathcal{R}_B \\
&\times \frac{\not p_a + m - \not k_2}{-2k_2p_a}\ \not\epsilon^\star_{\sigma_2} u(p_a,\lambda_a) \Bigg). \qquad (20)
\end{aligned}
$$

As in the previous case the expression for $L^{k_1,k_2}_{e^-,e^+}$ can be easily separated into parts:

$$
L^{k_1,k_2}_{e^-,e^+} = L^{k_1,k_2}_{e^-,e^+}(1) + L^{k_1,k_2}_{e^-,e^+}(\bar{2}) + L^{k_1,k_2}_{e^-,e^+}(\bar{3}). \qquad (21)
$$

The first part, $L^{k_1,k_2}_{e^-,e^+}(1)$, is gauge invariant by construction. It is also the only part from this group of diagrams with a numerator proportional both to the momenta of $k_1$ and $k_2$:

$$
\begin{aligned}
L^{k_1,k_2}_{e^-,e^+}(1) &= (eQ_e)^2 \\
&\times \Bigg( \bar{v}(p_b,\lambda_b)\ \not\epsilon^\star_{\sigma_2}(k_2) \frac{\not k_2}{-2k_2p_b} \mathcal{R}_B \\
&\times \frac{-\not k_1}{-2k_1p_a}\ \not\epsilon^\star_{\sigma_1}(k_1) u(p_a,\lambda_a) \\
&+ \bar{v}(p_b,\lambda_b)\ \not\epsilon^\star_{\sigma_1}(k_1) \frac{\not k_1}{-2k_1p_b} \mathcal{R}_B \\
&\times \frac{-\not k_2}{-2k_2p_a}\ \not\epsilon^\star_{\sigma_2}(k_2) u(p_a,\lambda_a) \Bigg). \qquad (22)
\end{aligned}
$$

The second term, $L^{k_1,k_2}_{e^-,e^+}(\bar{2})$, has two contributions, numerators linear either in $k_1$ or $k_2$; it reads

$$
\begin{aligned}
L^{k_1,k_2}_{e^-,e^+}(\bar{2}) &= (eQ_e)^2 \\
&\times \Bigg( \bar{v}(p_b,\lambda_b)\ \not\epsilon^\star_{\sigma_2}(k_2) \frac{-\not p_b + m}{-2k_2p_b} \mathcal{R}_B \\
&\times \frac{-\not k_1}{-2k_1p_a}\ \not\epsilon^\star_{\sigma_1}(k_1) u(p_a,\lambda_a) \\
&+ \bar{v}(p_b,\lambda_b)\ \not\epsilon^\star_{\sigma_1}(k_1) \frac{\not k_1}{-2k_1p_b} \mathcal{R}_B \\
&\times \frac{\not p_a + m}{-2k_2p_a}\ \not\epsilon^\star_{\sigma_2}(k_2) u(p_a,\lambda_a) \\
&+ \bar{v}(p_b,\lambda_b)\ \not\epsilon^\star_{\sigma_2}(k_2) \frac{\not k_2}{-2k_2p_b} \mathcal{R}_B \\
&\times \frac{\not p_a + m}{-2k_1p_a}\ \not\epsilon^\star_{\sigma_1}(k_1) u(p_a,\lambda_a) \\
&+ \bar{v}(p_b,\lambda_b)\ \not\epsilon^\star_{\sigma_1}(k_1) \frac{-\not p_b + m}{-2k_1p_b} \mathcal{R}_B \\
&\times \frac{-\not k_2}{-2k_2p_a}\ \not\epsilon^\star_{\sigma_2}(k_2) u(p_a,\lambda_a) \Bigg). \qquad (23)
\end{aligned}
$$

Finally the third one, $L^{k_1,k_2}_{e^-,e^+}(\bar{3})$, is free of both $k_1$ and $k_2$ in the numerator:

$$
\begin{aligned}
L^{k_1,k_2}_{e^-,e^+}(\bar{3}) &= (eQ_e)^2 \\
&\times \Bigg( \bar{v}(p_b,\lambda_b)\ \not\epsilon^\star_{\sigma_2}(k_2) \frac{-\not p_b + m}{-2k_2p_b} \mathcal{R}_B \\
&\times \frac{\not p_a + m}{-2k_1p_a}\ \not\epsilon^\star_{\sigma_1}(k_1) u(p_a,\lambda_a) \\
&+ \bar{v}(p_b,\lambda_b)\ \not\epsilon^\star_{\sigma_1}(k_1) \frac{-\not p_b + m}{-2k_1p_b} \mathcal{R}_B \\
&\times \frac{\not p_a + m}{-2k_2p_a}\ \not\epsilon^\star_{\sigma_2}(k_2) u(p_a,\lambda_a) \Bigg). \qquad (24)
\end{aligned}
$$

To complete the sub-set of diagrams for double bremsstrahlung from the initial state, the contribution of the double emission from a positron line should be added. We will omit the explicit formula analogous to (15) here, and the expressions for $L^{k_1,k_2}_{e^+}(1)$, $L^{k_1,k_2}_{e^+}(2)$, $L^{k_1,k_2}_{e^+}(\bar{3})$, $L^{k_1,k_2}_{e^+}(\bar{4})$. They can be obtained from $L^{k_1,k_2}_{e^-}(1)$, $L^{k_1,k_2}_{e^-}(2)$, $L^{k_1,k_2}_{e^-}(\bar{3})$, $L^{k_1,k_2}_{e^-}(\bar{4})$ by analogy, or by explicit calculation.

### 3.1 Diagrams with $Z$ exchange

Before going to the more complex case of $W$ exchange, where complications due to the $t$ dependence of the $W$ propagator occur, let us concentrate on $Z$ exchange. The diagrams discussed so far represent then the complete gauge invariant amplitude for the process $e^+e^- \to \nu_\mu\bar{\nu}_\mu\gamma\gamma$. In such a subgroup of diagrams (for the process $e^+e^- \to \nu_e\bar{\nu}_e\gamma\gamma$) the symbol $\mathcal{R}_{B=Z}$ always represents

$$
\begin{aligned}
\mathcal{R}_Z &= \left(\gamma^\mu(v\mathbf{1} + a\gamma^5)\right)_{\alpha\beta} \\
&\times \left(\bar{u}(p_c,\lambda_c)\gamma_\mu(v\mathbf{1} + a\gamma^5)v(p_d,\lambda_d)\right) \\
&\times BW_Z((p_c+p_d)^2), \qquad (25)
\end{aligned}
$$

which is a constant algebraic expression, independent of photon momenta and identical for all diagrams. The $Z$ boson propagator $BW_Z((p_c+p_d)^2)$ depends on the invariant mass of the outgoing neutrinos only. The bi-spinor indices of $\gamma^\mu$ and $\gamma^\mu\gamma^5$ matrices which enter into the matrix products of formulae such as (14) to (24) are explicitly given and denoted as $\alpha\beta$. The complete amplitude reads

$$
\begin{aligned}
\mathcal{M} &= L^{k_1,k_2}_{e^-} + L^{k_1,k_2}_{e^+} + L^{k_1,k_2}_{e^-,e^+} \\
&= L^{k_1,k_2}_{e^-}(1) + L^{k_1,k_2}_{e^-}(2) + L^{k_1,k_2}_{e^-}(\bar{3}) + L^{k_1,k_2}_{e^-}(\bar{4})
\end{aligned}
$$

$$+L_{e^+}^{k_1,k_2}(1) + L_{e^+}^{k_1,k_2}(2) + L_{e^+}^{k_1,k_2}(\bar{3}) + L_{e^+}^{k_1,k_2}(\bar{4})$$
$$+L_{e^-,e^+}^{k_1,k_2}(1) + L_{e^-,e^+}^{k_1,k_2}(\bar{2}) + L_{e^-,e^+}^{k_1,k_2}(\bar{3}). \qquad (26)$$

Here $L_{e^-}^{k_1,k_2}$ is given by (14) (or by (15) if separated into parts) and $L_{e^-,e^+}^{k_1,k_2}$ by (20) (or (21)). For $L_{e^+}^{k_1,k_2}$, the expressions of $L_{e^-}^{k_1,k_2}$ can be used with the appropriate changes of signs, momenta, etc.

The formula for the complete spin amplitude ($Z$ exchange only) can be easily re-ordered into consecutive contributions $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \ldots$, each gauge invariant and each representing an individual $L$, or group of $L$'s, in the square brackets:

$$\begin{aligned}
\mathcal{M} &= \mathfrak{M}^{Z}_{2\{\mathrm{I}\}}\left(\begin{smallmatrix} p\,k_1\,k_2 \\ \lambda\,\sigma_1\,\sigma_2 \end{smallmatrix}\right) \\
&= \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4 + \mathcal{M}_5 + \mathcal{M}_6 + \mathcal{M}_7 \\
&= L_{e^-}^{k_1,k_2}(1) + L_{e^-}^{k_1,k_2}(2) + L_{e^+}^{k_1,k_2}(1) + L_{e^+}^{k_1,k_2}(2) \\
&\quad +L_{e^-,e^+}^{k_1,k_2}(1) \\
&\quad + \left[L_{e^-}^{k_1,k_2}(\bar{4}) + L_{e^+}^{k_1,k_2}(\bar{4}) + L_{e^-,e^+}^{k_1,k_2}(\bar{2})\right] \\
&\quad + \left[L_{e^-}^{k_1,k_2}(\bar{3}) + L_{e^+}^{k_1,k_2}(\bar{3}) + L_{e^-,e^+}^{k_1,k_2}(\bar{3})\right]. \qquad (27)
\end{aligned}$$

As one can see, the sum of terms $\mathcal{M}_1$ to $\mathcal{M}_5$, contributing to $\beta_2^2$ of the CEEX exponentiation scheme (these terms are not infrared singular at all) is gauge invariant and clearly separated from the rest. It can be sliced further into *five* parts, each individually gauge invariant. The last two terms, $\mathcal{M}_6$ and $\mathcal{M}_7$ correspond respectively to $\beta_1^1$ and $\beta_0^0$ (multiplied by one or two soft photon factors) and can be obtained from a lower order of the perturbation expansion. It is rather straightforward to see that the term $\mathcal{M}_7$ consists of a Born level amplitude multiplied by soft factors corresponding to the emission of two photons. The term $\mathcal{M}_6$ consists of products: a soft factor for one of the photons and $\beta_1^1$ for the other one; see [8] for definitions. To see it better, it is convenient to order the expression accordingly to terms proportional either to $\not{k}_1$ or $\not{k}_2$.

Note also that for each of the parts to be gauge invariant, it is not necessary that four-momentum conservation is fulfilled. That is why the separation is easily adaptable to the extrapolation procedure used in KKMC [7]. Here we finish our discussion of results for $s$-channel exchange of $Z$. Let us now turn to the contributions related to the $t$-channel $W$ exchange.

### 3.2 Diagrams with $W$ exchange

First, let us note that all formulae presented so far are valid for the diagrams involving $W$ exchange as well. The difference is that instead of (25) for $\mathcal{R}_B$ one should use

$$\begin{aligned}
\mathcal{R}_W &= \left(\gamma_\mu(\mathbf{1} - \gamma^5)v(p_d,\lambda_d)\right)_\alpha \\
&\quad \times \left(\bar{u}(p_c,\lambda_c)\gamma^\mu(\mathbf{1} - \gamma^5)\right)_\beta BW_W(t). \qquad (28)
\end{aligned}$$

The spinorial form of this expression is universal, and, as in the case of $Z$ exchange, the same expression is to be used in all places. The difference lies in the $t$ dependence of the $W$ propagator; the transfer will depend on the way how the photon lines are attached to the fermionic ones. Nonetheless, in some groups of terms gauge cancellation occurs in the same way as before. If we recall the part of the $W$ exchange amplitude, written in analogy to (27) as

$$\begin{aligned}
\mathcal{M}^{\mathcal{A}}_{\mathcal{W}} &= \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4 + \mathcal{M}_5 + \bar{\mathcal{M}}_6 + \bar{\mathcal{M}}_7 \\
&= L_{e^-}^{k_1,k_2}(1) + L_{e^-}^{k_1,k_2}(2) + L_{e^+}^{k_1,k_2}(1) + L_{e^+}^{k_1,k_2}(2) \\
&\quad +L_{e^-,e^+}^{k_1,k_2}(1) \\
&\quad + \Big(L_{e^-}^{k_1,k_2}(\bar{4}) + L_{e^+}^{k_1,k_2}(\bar{4}) + L_{e^-,e^+}^{k_1,k_2}(\bar{2})\Big) \\
&\quad + \Big((L_{e^-}^{k_1,k_2}(\bar{3}) + L_{e^+}^{k_1,k_2}(\bar{3}) + L_{e^-,e^+}^{k_1,k_2}(\bar{3})\Big), \qquad (29)
\end{aligned}$$

then the parts $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$, $\mathcal{M}_4$ and $\mathcal{M}_5$ remain gauge invariant. Only the last two terms will need contributions from diagrams with triple and quartic gauge boson couplings for the gauge invariance to hold. To visualize this point, the bar sign is now placed over $\bar{\mathcal{M}}_6$ and $\bar{\mathcal{M}}_7$. Note that, as already pointed in the previous subsection, these are the contributions that could be obtained from the results of the calculation at lower perturbative order if the complications due to the variation of $W$ exchange transfers were not taken into account.

Let us continue with the second part of our discussion now. The two (now gauge dependent) contributions $\bar{\mathcal{M}}_6$ and $\bar{\mathcal{M}}_7$ will be completed first with diagrams from the left-hand side of Fig. 4. Note that these diagrams are the last ones with a photon line attached to an incoming electron/positron, thus the last ones contributing collinear and/or soft singularities. The contribution to the scattering amplitude from these new diagrams reads

$$\begin{aligned}
L_{e^-,W}^{k_1,k_2} &= (eQ_e)^2 \\
&\times BW_W\left((p_c + k_2 - p_a)^2\right) \\
&\times BW_W\left((p_c + k_2 + k_1 - p_a)^2\right) \\
&\times \left(\bar{v}(p_b,\lambda_b)(\mathbf{1} - \gamma^5)\gamma^\mu v(p_d,\lambda_d)\right. \\
&\times \left[g_{\mu\nu}(p-q)_\rho + g_{\nu\rho}(q-k_1)_\mu + g_{\mu\rho}(k_1-p)_\nu\right] \\
&\times \left(\epsilon^\star_{\sigma_1}(k_1)\right)^\rho \\
&\times \bar{u}(p_c,\lambda_c)(\mathbf{1} - \gamma^5)\gamma^\nu \frac{\not{p}_a + m - \not{k}_2}{-2k_2p_a} \\
&\times \not{\epsilon}^\star_{\sigma_2}(k_2)u(p_a,\lambda_a)\ \Big) \\
&+(1 \leftrightarrow 2). \qquad (30)
\end{aligned}$$

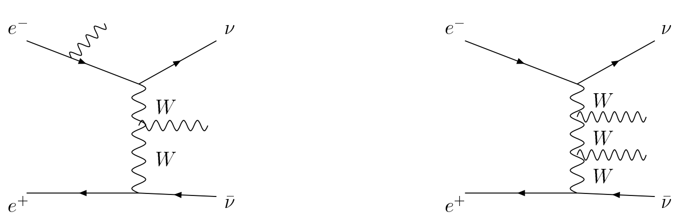


**Fig. 4.** Single and double emission from $W$

Here $p = p_d - p_b = -(p_c - p_a + k_1 + k_2)$ and $q = p_c - p_a + k_2 = -(p_d - p_b + k_1)$. Similarly one can write the contribution $L^{k_1,k_2}_{e^+,W}$ for the two diagrams with emission from positron and $W$, but we will omit the corresponding formulae. As before, we separate $L^{k_1,k_2}_{e^-,W} = L^{k_1,k_2}_{e^-,W}(k^0) + L^{k_1,k_2}_{e^-,W}(k^1)$ into parts, $(k^1)$ marks the contribution where only $\not{k}_2$ is taken from the fermionic propagator and $(k^0)$ marks the rest. The explicit formulae are

$$\begin{aligned}
L^{k_1,k_2}_{e^-,W}(k^0) &= (eQ_e)^2 \\
&\times BW_W\left((p_c + k_2 - p_a)^2\right) \\
&\times BW_W\left((p_c + k_2 + k_1 - p_a)^2\right) \\
&\times \Bigg(\bar{v}(p_b,\lambda_b)(\mathbf{1} - \gamma^5)\gamma^\mu v(p_d,\lambda_d) \\
&\times \left[g_{\mu\nu}(p-q)_\rho + g_{\nu\rho}(q-k_1)_\mu + g_{\mu\rho}(k_1-p)_\nu\right] \\
&\times \left(\epsilon^\star_{\sigma_1}(k_1)\right)^\rho \\
&\times \bar{u}(p_c,\lambda_c)(\mathbf{1} - \gamma^5)\gamma^\nu \frac{\not{p}_a + m}{-2k_2 p_a} \not{\epsilon}^\star_{\sigma_2}(k_2) u(p_a,\lambda_a)\Bigg) \\
&+ (1 \leftrightarrow 2)
\end{aligned} \tag{31}$$

and

$$\begin{aligned}
L^{k_1,k_2}_{e^-,W}(k^1) &= (eQ_e)^2 \\
&\times BW_W\left((p_c + k_2 - p_a)^2\right) \\
&\times BW_W\left((p_c + k_2 + k_1 - p_a)^2\right) \\
&\times \Bigg(\bar{v}(p_b,\lambda_b)(\mathbf{1} - \gamma^5)\gamma^\mu v(p_d,\lambda_d) \\
&\times \left[g_{\mu\nu}(p-q)_\rho + g_{\nu\rho}(q-k_1)_\mu + g_{\mu\rho}(k_1-p)_\nu\right] \\
&\times \left(\epsilon^\star_{\sigma_1}(k_1)\right)^\rho \\
&\times \bar{u}(p_c,\lambda_c)(\mathbf{1} - \gamma^5)\gamma^\nu \frac{-\not{k}_2}{-2k_2 p_a} \not{\epsilon}^\star_{\sigma_2}(k_2) u(p_a,\lambda_a)\Bigg) \\
&+ (1 \leftrightarrow 2).
\end{aligned} \tag{32}$$

Let us start with the second one, which can be easily transformed (with the help of the Dirac equation) into

$$\begin{aligned}
L^{k_1,k_2}_{e^-,W}(k^1) &= (eQ_e)^2 \\
&\times BW_W\left((p_c + k_2 - p_a)^2\right) \\
&\times BW_W\left((p_c + k_2 + k_1 - p_a)^2\right) \\
&\times \Bigg(\bar{v}(p_b,\lambda_b)(\mathbf{1} - \gamma^5)\gamma^\mu v(p_d,\lambda_d) \\
&\times \left[g_{\mu\nu}(p_d - p_b - p_c + p_a - k_2)_\rho + g_{\nu\rho}(-2k_1)_\mu\right. \\
&\left. + g_{\mu\rho}(2k_1 - p_a)_\nu\right] \left(\epsilon^\star_{\sigma_1}(k_1)\right)^\rho \\
&\times \bar{u}(p_c,\lambda_c)(\mathbf{1} - \gamma^5)\gamma^\nu \frac{-\not{k}_2}{-2k_2 p_a} \not{\epsilon}^\star_{\sigma_2}(k_2) u(p_a,\lambda_a)\Bigg) \\
&+ (1 \leftrightarrow 2).
\end{aligned} \tag{33}$$

This contribution can be separated even further:

$$L^{k_1,k_2}_{e^-,W}(k^1) = L^{k_1,k_2}_{e^-,W}(\bar{1}) + L^{k_1,k_2}_{e^-,W}(2) + L^{k_1,k_2}_{e^-,W}(\bar{3}), \tag{34}$$

where

$$\begin{aligned}
L^{k_1,k_2}_{e^-,W}(\bar{1}) &= (eQ_e)^2 \\
&\times BW_W\left((p_c + k_2 - p_a)^2\right) \\
&\times BW_W\left((p_c + k_2 + k_1 - p_a)^2\right) \\
&\times \Bigg(\bar{v}(p_b,\lambda_b)(\mathbf{1} - \gamma^5)\gamma^\mu v(p_d,\lambda_d) \\
&\times \bar{u}(p_c,\lambda_c)(\mathbf{1} - \gamma^5)\gamma^\mu \frac{-\not{k}_2}{-2k_2 p_a} \not{\epsilon}^\star_{\sigma_2}(k_2) u(p_a,\lambda_a)\Bigg) \\
&\times (p_d - p_b - p_c + p_a - k_2) \cdot \epsilon^\star_{\sigma_1}(k_1) \\
&+ (1 \leftrightarrow 2),
\end{aligned} \tag{35}$$

and the second term

$$\begin{aligned}
L^{k_1,k_2}_{e^-,W}(2) &= (eQ_e)^2 \\
&\times BW_W\left((p_c + k_2 - p_a)^2\right) \\
&\times BW_W\left((p_c + k_2 + k_1 - p_a)^2\right) \\
&\times \Bigg(\bar{v}(p_b,\lambda_b)(\mathbf{1} - \gamma^5)\gamma^\mu v(p_d,\lambda_d) \\
&\times 2\left[-\left(\epsilon^\star_{\sigma_1}(k_1)\right)_\nu (k_1)_\mu + \left(\epsilon^\star_{\sigma_1}(k_1)\right)_\mu (k_1)_\nu\right] \\
&\times \bar{u}(p_c,\lambda_c)(\mathbf{1} - \gamma^5)\gamma^\nu \frac{-\not{k}_2}{-2k_2 p_a} \not{\epsilon}^\star_{\sigma_2}(k_2) u(p_a,\lambda_a)\Bigg) \\
&+ (1 \leftrightarrow 2)
\end{aligned} \tag{36}$$

is gauge invariant by itself. The third one is less divergent in the collinear configuration:

$$\begin{aligned}
L^{k_1,k_2}_{e^-,W}(\bar{3}) &= -(eQ_e)^2 \\
&\times BW_W\left((p_c + k_2 - p_a)^2\right) \\
&\times BW_W\left((p_c + k_2 + k_1 - p_a)^2\right) \\
&\times \Bigg(\bar{v}(p_b,\lambda_b)(\mathbf{1} - \gamma^5) \not{\epsilon}^\star_{\sigma_1}(k_1) v(p_d,\lambda_d) \\
&\times \bar{u}(p_c,\lambda_c)(\mathbf{1} - \gamma^5)\not{p}_a \frac{-\not{k}_2}{-2k_2 p_a} \not{\epsilon}^\star_{\sigma_2}(k_2) u(p_a,\lambda_a)\Bigg) \\
&+ (1 \leftrightarrow 2).
\end{aligned} \tag{37}$$

Let us now turn to the other part of the amplitude and present it in the form of a sum:

$$L^{k_1,k_2}_{e^-,W}(k^0) = L^{k_1,k_2}_{e^-,W}(\bar{4}) + L^{k_1,k_2}_{e^-,W}(5) + L^{k_1,k_2}_{e^-,W}(\bar{6}), \tag{38}$$

where the first term reads

$$\begin{aligned}
L^{k_1,k_2}_{e^-,W}(\bar{4}) &= (eQ_e)^2 \\
&\times BW_W\left((p_c + k_2 - p_a)^2\right)
\end{aligned}$$

$$\times BW_W\left((p_c+k_2+k_1-p_a)^2\right)$$
$$\times\bigg(\bar v(p_b,\lambda_b)(\mathbf{1}-\gamma^5)\gamma^\mu v(p_d,\lambda_d)$$
$$\times\,\bar u(p_c,\lambda_c)(\mathbf{1}-\gamma^5)\gamma^\mu\frac{\not p_a+m}{-2k_2p_a}\;\not\epsilon^\star_{\sigma_2}(k_2)u(p_a,\lambda_a)\bigg)$$
$$\times(p_d-p_b-p_c+p_a-k_2)\cdot\epsilon^\star_{\sigma_1}(k_1)$$
$$+(1\leftrightarrow 2), \tag{39}$$

the second one

$$L^{k_1,k_2}_{e^-,W}(5)=(eQ_e)^2$$
$$\times BW_W\left((p_c+k_2-p_a)^2\right)$$
$$\times BW_W\left((p_c+k_2+k_1-p_a)^2\right)$$
$$\times\bigg(\bar v(p_b,\lambda_b)(\mathbf{1}-\gamma^5)\gamma^\mu v(p_d,\lambda_d)$$
$$\times 2\left[-\left(\epsilon^\star_{\sigma_1}(k_1)\right)_\nu((k_1)_\mu-(p_b)_\mu)\right.$$
$$\left.+\left(\epsilon^\star_{\sigma_1}(k_1)\right)_\mu(k_1)_\nu\right]$$
$$\times\,\bar u(p_c,\lambda_c)(\mathbf{1}-\gamma^5)\gamma^\nu\frac{\not p_a+m}{-2k_2p_a}\;\not\epsilon^\star_{\sigma_2}(k_2)u(p_a,\lambda_a)\bigg)$$
$$+(1\leftrightarrow 2), \tag{40}$$

and the third one

$$L^{k_1,k_2}_{e^-,W}(\bar 6)=(eQ_e)^2$$
$$\times BW_W\left((p_c+k_2-p_a)^2\right)$$
$$\times BW_W\left((p_c+k_2+k_1-p_a)^2\right)$$
$$\times\bigg(\bar v(p_b,\lambda_b)(\mathbf{1}-\gamma^5)\;\not\epsilon^\star_{\sigma_1}(k_1)v(p_d,\lambda_d)$$
$$\times\bar u(p_c,\lambda_c)(\mathbf{1}-\gamma^5)(\not k_2-\not p_a)$$
$$\times\,\frac{\not p_a+m}{-2k_2p_a}\;\not\epsilon^\star_{\sigma_2}(k_2)u(p_a,\lambda_a)\bigg)$$
$$+(1\leftrightarrow 2). \tag{41}$$

The last two terms can be modified further and some terms neglected. One can check that these terms contribute at the level of $\frac{m_e}{\sqrt{s}}$ only, and that is why we will exclude them from explicit considerations. After these simplifications, we finally obtain, gauge invariant by itself,

$$L^{k_1,k_2}_{e^-,W}(5)=(eQ_e)^2$$
$$\times BW_W\left((p_c+k_2-p_a)^2\right)$$
$$\times BW_W\left((p_c+k_2+k_1-p_a)^2\right)$$
$$\times\bigg(\bar v(p_b,\lambda_b)(\mathbf{1}-\gamma^5)\gamma^\mu v(p_d,\lambda_d)$$
$$\times 2\left[-\left(\epsilon^\star_{\sigma_1}(k_1)\right)_\nu(k_1)_\mu+\left(\epsilon^\star_{\sigma_1}(k_1)\right)_\mu(k_1)_\nu\right]$$
$$\times\,\bar u(p_c,\lambda_c)(\mathbf{1}-\gamma^5)\gamma^\nu\frac{\not p_a+m}{-2k_2p_a}\;\not\epsilon^\star_{\sigma_2}(k_2)u(p_a,\lambda_a)\bigg)$$
$$+(1\leftrightarrow 2), \tag{42}$$

and now we have the term explicitly less divergent in the collinear configuration

$$L^{k_1,k_2}_{e^-,W}(\bar 6)=(eQ_e)^2$$
$$\times BW_W\left((p_c+k_2-p_a)^2\right)$$
$$\times BW_W\left((p_c+k_2+k_1-p_a)^2\right)$$
$$\times\bigg(\bar v(p_b,\lambda_b)(\mathbf{1}-\gamma^5)\;\not\epsilon^\star_{\sigma_1}(k_1)v(p_d,\lambda_d)$$
$$\times\,\bar u(p_c,\lambda_c)(\mathbf{1}-\gamma^5)\not k_2\frac{\not p_a+m}{-2k_2p_a}\;\not\epsilon^\star_{\sigma_2}(k_2)u(p_a,\lambda_a)\bigg)$$
$$+(1\leftrightarrow 2). \tag{43}$$

Let us now turn to the diagrams of double emission from the $W$; see the right-hand side of Fig. 4. The corresponding amplitude can be written as

$$L^{k_1,k_2}_{W,W}=(eQ_e)^2$$
$$\times BW_W\left((p_c-p_a)^2\right)BW_W\left((p_c+k_1-p_a)^2\right)$$
$$\times BW_W\left((p_c+k_1+k_2-p_a)^2\right)$$
$$\times\left(\bar v(p_b,\lambda_b)(\mathbf{1}-\gamma^5)\gamma^\mu v(p_d,\lambda_d)\right.$$
$$\left.\times\,\bar u(p_c,\lambda_c)(\mathbf{1}-\gamma^5)\gamma^\nu u(p_a,\lambda_a)\right)$$
$$\times\left[g_{\sigma\nu}(p-q)_\rho+g_{\nu\rho}(q-k_1)_\sigma+g_{\sigma\rho}(k_1-p)_\nu\right]$$
$$\times\left[g^\sigma_\mu(p'-q')_{\rho'}+g^\sigma_{\rho'}(q'-k_2)_\mu+g_{\mu\rho'}(k_2-p')^\sigma\right]$$
$$\times(\epsilon^\star_{\sigma_1}(k_1))^\rho(\epsilon^\star_{\sigma_2}(k_2))^{\rho'}$$
$$+(1\leftrightarrow 2), \tag{44}$$

where $p=-q'=p_d-p_b+k_2=-(p_c-p_a+k_1)$, $q=p_c-p_a=-(p_d-p_b+k_1+k_2)$ and $p'=p_d-p_b=-(p_c-p_a+k_1+k_2)$.

As usual, we will represent this expression in the form of a sum:

$$L^{k_1,k_2}_{W,W}=L^{k_1,k_2}_{W,W}(\bar 1)+L^{k_1,k_2}_{W,W}(\bar 2)+L^{k_1,k_2}_{W,W}(\bar 3)+L^{k_1,k_2}_{W,W}(\bar 4)$$
$$+L^{k_1,k_2}_{W,W}(\bar 5)+L^{k_1,k_2}_{W,W}(\bar 6). \tag{45}$$

The first term, proportional to a Born amplitude multiplied by a factor depending on the polarization of the two photons, takes the form

$$L^{k_1,k_2}_{W,W}(\bar 1)=(eQ_e)^2$$
$$\times BW_W\left((p_c-p_a)^2\right)BW_W\left((p_c+k_1-p_a)^2\right)$$
$$\times BW_W\left((p_c+k_1+k_2-p_a)^2\right)$$
$$\times\left(\bar v(p_b,\lambda_b)(\mathbf{1}-\gamma^5)\gamma^\mu v(p_d,\lambda_d)\right.$$
$$\left.\times\,\bar u(p_c,\lambda_c)(\mathbf{1}-\gamma^5)\gamma_\mu u(p_a,\lambda_a)\right)$$
$$\times(p-q)\cdot\epsilon^\star_{\sigma_1}(k_1)\;(p'-q')\cdot(\epsilon^\star_{\sigma_2}(k_2))$$
$$+(1\leftrightarrow 2). \tag{46}$$

Terms with a dependence on the polarization of only one photon factorize out from the amplitude and take the form

$$
\begin{aligned}
L^{k_1,k_2}_{W,W}(\bar{2}) &= (eQ_e)^2 \\
&\times BW_W\left((p_c-p_a)^2\right) BW_W\left((p_c+k_1-p_a)^2\right) \\
&\times BW_W\left((p_c+k_1+k_2-p_a)^2\right) \\
&\times \big(\bar{v}(p_b,\lambda_b)(\mathbf{1}-\gamma^5)\gamma^\mu v(p_d,\lambda_d) \\
&\times \bar{u}(p_c,\lambda_c)(\mathbf{1}-\gamma^5)\gamma^\nu u(p_a,\lambda_a)\big) \\
&\times 2\left[-(\epsilon^\star_{\sigma_1}(k_1))_\nu (k_1)_\mu + (k_1)_\nu (\epsilon^\star_{\sigma_1}(k_1))_\mu\right] \\
&\times (p'-q')\cdot(\epsilon^\star_{\sigma_2}(k_2)) \\
&+(1 \leftrightarrow\ 2)
\end{aligned}
\tag{47}
$$

and

$$
\begin{aligned}
L^{k_1,k_2}_{W,W}(\bar{3}) &= (eQ_e)^2 \\
&\times BW_W\left((p_c-p_a)^2\right) BW_W\left((p_c+k_1-p_a)^2\right) \\
&\times BW_W\left((p_c+k_1+k_2-p_a)^2\right) \\
&\times \big(\bar{v}(p_b,\lambda_b)(\mathbf{1}-\gamma^5)\gamma^\mu v(p_d,\lambda_d) \\
&\times \bar{u}(p_c,\lambda_c)(\mathbf{1}-\gamma^5)\gamma^\nu u(p_a,\lambda_a)\big) \\
&\times 2\left[-(\epsilon^\star_{\sigma_2}(k_2))_\nu (k_2)_\mu + (k_2)_\nu (\epsilon^\star_{\sigma_2}(k_2))_\mu\right] \\
&\times (p-q)\cdot\epsilon^\star_{\sigma_1}(k_1) \\
&+(1 \leftrightarrow 2).
\end{aligned}
\tag{48}
$$

The two last terms (47) and (48) are partially gauge independent, respectively for the polarization vector of the first and second photon. The remaining, fully gauge dependent parts of the amplitude read

$$
\begin{aligned}
L^{k_1,k_2}_{W,W}(\bar{4}) &= -(eQ_e)^2 \\
&\times BW_W\left((p_c-p_a)^2\right) BW_W\left((p_c+k_1-p_a)^2\right) \\
&\times BW_W\left((p_c+k_1+k_2-p_a)^2\right) \\
&\times \big(\bar{v}(p_b,\lambda_b)(\mathbf{1}-\gamma^5)\not{k}_2 v(p_d,\lambda_d) \\
&\times \bar{u}(p_c,\lambda_c)(\mathbf{1}-\gamma^5)\ \not{\epsilon}^\star_{\sigma_1} u(p_a,\lambda_a)\big) \\
&\times (p'-q')\cdot(\epsilon^\star_{\sigma_2}(k_2)) \\
&+(1 \leftrightarrow\ 2)
\end{aligned}
\tag{49}
$$

and

$$
\begin{aligned}
L^{k_1,k_2}_{W,W}(\bar{5}) &= (eQ_e)^2 \\
&\times BW_W\left((p_c-p_a)^2\right) BW_W\left((p_c+k_1-p_a)^2\right) \\
&\times BW_W\left((p_c+k_1+k_2-p_a)^2\right) \\
&\times \big(\bar{v}(p_b,\lambda_b)(\mathbf{1}-\gamma^5)\ \not{\epsilon}^\star_{\sigma_2} v(p_d,\lambda_d) \\
&\times \bar{u}(p_c,\lambda_c)(\mathbf{1}-\gamma^5)\not{k}_1 u(p_a,\lambda_a)\big) \\
&\times (p-q)\cdot\epsilon^\star_{\sigma_1}(k_1) + (1 \leftrightarrow 2).
\end{aligned}
\tag{50}
$$

Finally


**Fig. 5.** Four boson coupling and coupling for unphysical $\chi$ field

$$
\begin{aligned}
L^{k_1,k_2}_{W,W}(\bar{6}) &= (eQ_e)^2 \\
&\times BW_W\left((p_c-p_a)^2\right) BW_W\left((p_c+k_1-p_a)^2\right) \\
&\times BW_W\left((p_c+k_1+k_2-p_a)^2\right) \\
&\times \big(\bar{v}(p_b,\lambda_b)(\mathbf{1}-\gamma^5)\gamma^\mu v(p_d,\lambda_d) \\
&\times \bar{u}(p_c,\lambda_c)(\mathbf{1}-\gamma^5)\gamma^\nu u(p_a,\lambda_a)\big) \\
&\times \left[g_{\nu\rho}(q-k_1)_\sigma + g_{\sigma\rho}(k_1-p)_\nu\right](\epsilon^\star_{\sigma_1}(k_1))^\rho \\
&\times \left[g^\sigma_{\rho'}(q'-k_2)_\mu + g_{\mu\rho'}(k_2-p')^\sigma\right](\epsilon^\star_{\sigma_2}(k_2))^{\rho'} \\
&+(1 \leftrightarrow 2).
\end{aligned}
\tag{51}
$$

As the last step, let us turn to contributions from the diagrams presented in Fig. 5. The diagram with the contribution from the quartic gauge coupling reads

$$
\begin{aligned}
L^{k_1,k_2}_{W^2} &= (eQ_e)^2 \\
&\times BW_W\left((p_c-p_a)^2\right) BW_W\left((p_c+k_2+k_1-p_a)^2\right) \\
&\times \big(\bar{v}(p_b,\lambda_b)(1-\gamma^5)\ \not{\epsilon}^\star_{\sigma_1} v(p_d,\lambda_d)\bar{u}(p_c,\lambda_c)(1-\gamma^5) \\
&\times\ \not{\epsilon}^\star_{\sigma_2} u(p_a,\lambda_a) \\
&+\bar{v}(p_b,\lambda_b)(1-\gamma^5)\ \not{\epsilon}^\star_{\sigma_2} v(p_d,\lambda_d)\bar{u}(p_c,\lambda_c)(1-\gamma^5) \\
&\times\ \not{\epsilon}^\star_{\sigma_1} u(p_a,\lambda_a) \\
&+2\bar{v}(p_b,\lambda_b)(1-\gamma^5)\gamma^\mu v(p_d,\lambda_d) \\
&\times \bar{u}(p_c,\lambda_c)(1-\gamma^5)\gamma_\mu u(p_a,\lambda_a)\epsilon^\star_{\sigma_1}\cdot\epsilon^\star_{\sigma_2}\big)\,.
\end{aligned}
\tag{52}
$$

It is convenient to write it as a sum of two parts

$$
L^{k_1,k_2}_{W^2} = L^{k_1,k_2}_{W^2}(\bar{1}) + L^{k_1,k_2}_{W^2}(\bar{2}),
\tag{53}
$$

where

$$
\begin{aligned}
L^{k_1,k_2}_{W^2}(\bar{1}) &= 2(eQ_e)^2 \\
&\times BW_W\left((p_c-p_a)^2\right) BW_W\left((p_c+k_2+k_1-p_a)^2\right) \\
&\times \epsilon^\star_{\sigma_1}\epsilon^\star_{\sigma_2}\bar{v}(p_b,\lambda_b)(\mathbf{1}-\gamma^5)\gamma^\mu v(p_d,\lambda_d) \\
&\times \bar{u}(p_c,\lambda_c)(\mathbf{1}-\gamma^5)\gamma_\mu u(p_a,\lambda_a)
\end{aligned}
\tag{54}
$$

and

$$
\begin{aligned}
L^{k_1,k_2}_{W^2}(\bar{2}) &= (eQ_e)^2 \\
&\times BW_W\left((p_c-p_a)^2\right) BW_W\left((p_c+k_2+k_1-p_a)^2\right) \\
&\times \big(\bar{v}(p_b,\lambda_b)(\mathbf{1}-\gamma^5)\ \not{\epsilon}^\star_{\sigma_1} v(p_d,\lambda_d) \\
&\times \bar{u}(p_c,\lambda_c)(\mathbf{1}-\gamma^5)\ \not{\epsilon}^\star_{\sigma_2} u(p_a,\lambda_a) \\
&+\bar{v}(p_b,\lambda_b)(\mathbf{1}-\gamma^5)\ \not{\epsilon}^\star_{\sigma_2} v(p_d,\lambda_d) \\
&\times \bar{u}(p_c,\lambda_c)(\mathbf{1}-\gamma^5)\ \not{\epsilon}^\star_{\sigma_1} u(p_a,\lambda_a)\big)\,.
\end{aligned}
\tag{55}
$$

The contribution from the diagram involving an internal $\chi$ line reads

$$\begin{aligned} L^{k_1,k_2}_{W,\chi} &= (eQ_e)^2 M_W^2 \\ &\times BW_W\left((p_c-p_a)^2\right) BW_W\left((p_c+k_2+k_1-p_a)^2\right) \\ &\times \left(\bar{v}(p_b,\lambda_b)(\mathbf{1}-\gamma^5)\ \not{\epsilon}^{\star}_{\sigma_1}(k_1)v(p_d,\lambda_d)\right. \\ &\times \bar{u}(p_c,\lambda_c)(\mathbf{1}-\gamma^5)\ \left.\not{\epsilon}^{\star}_{\sigma_2}(k_2)u(p_a,\lambda_a)\right) \\ &+(1\leftrightarrow 2). \end{aligned} \tag{56}$$

This closes the list of all diagrams entering the complete spin amplitude for the process $e^+e^- \to \nu_e\bar{\nu}_e\gamma\gamma$. The contributing terms were obtained from the Feynman rules and were grouped on the basis of rather straightforward rules: gauge symmetry and the nature of the singularities in the infrared and collinear limits (phase space integration was not necessary).

The complete gauge invariant part of the spin amplitude of $W$ exchange can be written now as

$$\mathcal{M}_{\mathcal{W}} = \mathcal{M}^{\mathcal{A}}_{\mathcal{W}} + \mathcal{M}^{\mathcal{B}}_{\mathcal{W}}, \tag{57}$$

where $\mathcal{M}^{\mathcal{A}}_{\mathcal{W}}$ (technically identical to the amplitude of $Z$ exchange) was given by formula (29) and the new part, specific to the $W$ bosonic interactions, reads

$$\begin{aligned} \mathcal{M}^{\mathcal{B}}_{W} &= L^{k_1,k_2}_{e^+,W}(\bar{1}) + L^{k_1,k_2}_{e^+,W}(2) + L^{k_1,k_2}_{e^+,W}(\bar{3}) + L^{k_1,k_2}_{e^+,W}(\bar{4}) \\ &+L^{k_1,k_2}_{e^+,W}(5) + L^{k_1,k_2}_{e^+,W}(\bar{6}) \\ &+L^{k_1,k_2}_{e^-,W}(\bar{1}) + L^{k_1,k_2}_{e^-,W}(2) + L^{k_1,k_2}_{e^-,W}(\bar{3}) + L^{k_1,k_2}_{e^-,W}(\bar{4}) \\ &+L^{k_1,k_2}_{e^-,W}(5) + L^{k_1,k_2}_{e^-,W}(\bar{6}) \\ &+L^{k_1,k_2}_{W,W}(\bar{1}) + L^{k_1,k_2}_{W,W}(\bar{2}) + L^{k_1,k_2}_{W,W}(\bar{3}) + L^{k_1,k_2}_{W,W}(\bar{4}) \\ &+L^{k_1,k_2}_{W,W}(\bar{5}) + L^{k_1,k_2}_{W,W}(\bar{6}) \\ &+L^{k_1,k_2}_{W^2}(\bar{1}) + L^{k_1,k_2}_{W^2}(\bar{2}) + L^{k_1,k_2}_{W,\chi}. \end{aligned} \tag{58}$$

We can now write the complete spin amplitude of the $W$ interactions as a sum of gauge invariant parts:

$$\begin{aligned} \mathcal{M}_{\mathcal{W}} &= \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4 + \mathcal{M}_5 + \mathcal{M}_6 + \mathcal{M}_7 \\ &+\mathcal{M}_8 + \mathcal{M}_9 + \mathcal{M}_{10} + \mathcal{M}_{11}, \end{aligned} \tag{59}$$

where

$$\begin{aligned} \mathcal{M}_1 &= L^{k_1,k_2}_{e^-}(1), \\ \mathcal{M}_2 &= L^{k_1,k_2}_{e^-}(2), \\ \mathcal{M}_3 &= L^{k_1,k_2}_{e^+}(1), \\ \mathcal{M}_4 &= L^{k_1,k_2}_{e^+}(2), \\ \mathcal{M}_5 &= L^{k_1,k_2}_{e^-,e^+}(1), \\ \mathcal{M}_6 &= L^{k_1,k_2}_{e^-}(\bar{4}) + L^{k_1,k_2}_{e^+}(\bar{4}) + L^{k_1,k_2}_{e^-,e^+}(\bar{2}) + L^{k_1,k_2}_{e^-,W}(\bar{1}) \\ &+L^{k_1,k_2}_{e^+,W}(\bar{1}), \\ \mathcal{M}_7 &= L^{k_1,k_2}_{e^-}(\bar{3}) + L^{k_1,k_2}_{e^+}(\bar{3}) + L^{k_1,k_2}_{e^-,e^+}(\bar{3}) + L^{k_1,k_2}_{e^-,W}(\bar{4}) \\ &+L^{k_1,k_2}_{e^+,W}(\bar{4}) + L^{k_1,k_2}_{W,W}(\bar{1}) + L^{k_1,k_2}_{W^2}(\bar{1}), \\ \mathcal{M}_8 &= L^{k_1,k_2}_{e^-,W}(2), \\ \mathcal{M}_9 &= L^{k_1,k_2}_{e^+,W}(2), \\ \mathcal{M}_{10} &= L^{k_1,k_2}_{e^-,W}(5) + L^{k_1,k_2}_{e^+,W}(5) + L^{k_1,k_2}_{W,W}(\bar{2}) + L^{k_1,k_2}_{W,W}(\bar{3}), \\ \mathcal{M}_{11} &= L^{k_1,k_2}_{e^-,W}(\bar{3}) + L^{k_1,k_2}_{e^+,W}(\bar{3}) + L^{k_1,k_2}_{e^-,W}(\bar{6}) \\ &+L^{k_1,k_2}_{e^+,W}(\bar{6}) + L^{k_1,k_2}_{W,W}(\bar{4}) + L^{k_1,k_2}_{W,W}(\bar{5}) + L^{k_1,k_2}_{W,W}(\bar{6}) \\ &+L^{k_1,k_2}_{W^2}(\bar{2}) + L^{k_1,k_2}_{W,\chi}. \end{aligned} \tag{60}$$

The matching of the $L^a_b(\bar{n})$ terms into gauge invariant parts $\mathcal{M}_i$ of the amplitude is straightforward and based on the type of singularities present/absent in the particular group. Each of the contributions $\mathcal{M}_1$–$\mathcal{M}_{11}$ listed below can be given some physical interpretation. In some cases, the appearance of such parts may seem rather unexpected. In brackets we provide symbols such as (IA); they denote the name of the variables used in KKMC [7] Monte Carlo, as keys for the parts of the amplitude as listed here.

(1) $\mathcal{M}_1$ (IA), the contribution of double emission from an electron line (infrared non-singular part) with straightforward gauge cancellation within the terms originating from diagram of two photons attached to the same incoming electron line.

(2) $\mathcal{M}_2$ (IV2), the contribution of double emission from an electron line (infrared non-singular part) with non-straightforward gauge cancellation within the terms originating from diagram of two photons attached to the same incoming electron line. Part of the diagram's contribution had to be subtracted (more precisely, expressed without the $k_1k_2$ product in the electron propagator). This subtraction term is recuperated in $\mathcal{M}_6$ and $\mathcal{M}_7$.

(3) $\mathcal{M}_3$ (IA), $\mathcal{M}_4$ (IV1), the same as the previous two cases but for emission from a positron line.

(4) $\mathcal{M}_5$ (I8), the infrared non-singular contributions of single emission from an electron and another infrared non-singular part of single emission from a positron line. This contribution is gauge invariant by construction.

(5) $\mathcal{M}_6$ (I9X), (I9Y), (I9Z), (I9T), part of the amplitude with an infrared factor for one photon, and an infrared non-singular gauge invariant contribution for the second one. For the diagrams with $W$ exchange, the contribution from the diagram with photon emission from $W$ needs to be added. For the gauge cancellation to hold, the relation between $t$-channel transfers in the $W$ propagators and momenta multiplying the photon polarization vector needs to be fulfilled. Nonetheless, a certain freedom of choice is left. This freedom was useful in the construction of the extrapolation procedures[10].

(6) $\mathcal{M}_7$ (IVI), the part of the amplitude with infrared factors for both photons. For the diagrams with the $W$ exchange contribution from diagrams with single and double

[10] An identical condition, also originating directly from Ward identities, needs to be preserved in $\mathcal{M}_{10}$ and a similar one in $\mathcal{M}_{11}$.

emission of photons from $W$ needs to be taken, and also the part of the diagram with a quartic gauge coupling was needed here.
(7) $\mathcal{M}_8$ (I71), (I72), the part of the amplitude with an infrared non-singular contribution of emission from an electron for one photon and for another one part of emission from $W$ which is self gauge-conserving.
(8) $\mathcal{M}_9$ (I71), (I72), the same as the previous case but for emission from positron.
(9) $\mathcal{M}_{10}$ (I9s1), (I9s2), the part of the amplitude with an infrared factor for one photon and part of the emission from $W$, which is self gauge-conserving for another one.
(10) $\mathcal{M}_{11}$ (I9), (I9B), (I10), all the remaining parts; they turn out to be free of singularities both in collinear and soft limits.

We note the following.
(1) Let us comment that in the limit $M_W \to \infty$ all contributions from $\mathcal{M}_6$ to $\mathcal{M}_{11}$ disappear. In this limit amplitudes for $s$-channel $Z$ exchange and $t$-channel $W$ nearly coincide. The only remaining difference is the coupling constants and the hard interaction part of the amplitude given respectively by (25) and (28). This is an extension of the similar observation of [14], instrumental in the construction of extrapolation procedures of [10] to the case beyond real photon interactions with fermions only.
(2) Let us point out that in many places we have used a separation of the $WW\gamma$ vertex into three parts:
(i) the one with the $g_{\mu\nu}$ tensor along a line connecting the fermion lines;
(ii) the part internally preserving gauge symmetry, and
(iii) the remaining part which we often could reduce significantly with the help of the Dirac equation (because of the fermion lines connected with the $WW\gamma$ vertex by the $W$ propagator).
(3) Finally let us note that the above separation into gauge invariant parts can be continued even further. For example, it is rather easy to separate $\mathcal{M}_6$ into four parts. For each, the emissions of individual photons are attributed *either* to an electron *or* a positron line.

We have not exploited to the end the properties of $\mathcal{M}_{11}$. It was not interesting from the point of view of our main purpose, which is implementation of the matrix element to the environment of coherent exlusive exponentiation. Also in case of $\mathcal{M}_{11}$, and contrary to all the other parts from $\mathcal{M}_1$ to $\mathcal{M}_{10}$, similarities with first-order results could not be seen. This is rather natural, as for example quartic gauge couplings are absent in first order. In this case, hints of a pattern for constructing amplitudes of even higher order using iteration techniques could not be found. To this end, a discussion of the amplitudes of triple photon emission would be needed. If conclusive, it would point to solutions helpful beyond next-to-leading-log approximation, and thus beyond the immediate interest of the present paper.

Gauge invariance was not the only criterion which was used to split amplitude into parts. Equally important was that the two main sources of the radiation (incoming beams) form the unambiguous frame. In this frame photons' energies and the angles between photons and fermions could be defined[11]. That is why there was no need to make any reference to the regulators. Singular terms could be localized already at the amplitude level and in a fully differential manner, with no need to partially integrate phase space. The expansion in the contact interaction for the $W$ propagator allows the gauge cancellation effects of emission from $t$-channel $W$ to be placed within the frame of ISR radiation. Also, the relation between amplitude for double- and single-photon emission had to be exploited to remove ambiguities. Once these assumptions and properties were exploited, the solution seemed to be unique, up to a possible grouping or further splitting of the obtained parts. Confirmation of whether this is an accidental property which holds for this particular case (and up to a second order only) may require calculation of at least third order and for other processes as well.

## 4 Some points on extrapolation

Let us summarize here some specific issues related to the extrapolation procedure of a CEEX scheme described only for purely $s$-channel hard process in detail in [8]. One of the important properties of the perturbation expansion, rearranged to improve convergence into exclusive exponentiation, is that parts of the amplitudes need to be appropriately shifted between the orders of expansion. We will concentrate our attention on issues related to real bremsstrahlung only. Those parts of the higher-order terms (directly calculated in a standard way), which are already included at lower level of the CEEX perturbation expansion, need to be localized and subtracted from Feynman diagram calculation in a clear way. Only the remaining residual parts, called $\beta^0$, $\beta^1$, $\beta^2$ etc. [2], will be the higher-order terms. The use of $\beta$ functions is unambiguous if a sufficiently high order of perturbative calculation is available. However, this is not always the case; practical solutions for exponentiation require the definition of methods to calculate matrix elements for the kinematical configuration with a large number of real photons, when using results of the first (or second) order of the perturbation expansion only.

There are several rules which the extrapolation procedure must fulfill. Already the lowest order must include all terms with the highest power of the infrared singularity and all kinematical configurations for an arbitrary number of real photons in a fully exclusive manner. Then the first order provides all terms with the next to highest power of the infrared singularity, etc. Let us stress that the reduction/extrapolation procedure of exponentiation offers some freedom of choice. This freedom can be used to further improve the convergence of the perturbation expansion. The best guidance is of course comparison with the result of an even higher order of expansion, to minimize its

[11] The definition of this frame is process independent and helps to fix the gauge for amplitudes of single and double photon emission in a consistent manner, also in cases when extrapolation procedures are needed. Note that this point would require more elaboration if we were interested in the properties of the amplitudes for final-state radiation.

contribution. If such results are not available, higher-order leading-log results (for partially inclusive quantities) can be used instead. Finally, let us stress that if a sufficiently high order of the perturbation expansion is available, the dependence on the particular choice of extrapolation drops out and a unique result, identical to the one of the direct perturbation expansion without any reordering is obtained. Unfortunately this is not expected to be the case in the foreseeable future.

In the case of diagrams with $Z$ exchange, the choice of the extrapolation procedure is straightforward. Inspection of first-order (see (2)) and second-order (see (27)) amplitudes points to the following solution: the terms $\mathcal{M}_1$ to $\mathcal{M}_5$ of (27) should contribute to $\beta^2$, whereas the last two terms $\mathcal{M}_6$ and $\mathcal{M}_7$ can be directly obtained from the lower order. The $\mathcal{M}_6$ can be obtained from $\beta^1$ by multiplication by the soft photon factor for the other photon. The $\beta^1$ can be identified as this part of $\mathcal{M}^0$ (see (2)), which is proportional to $\not{k}_1$. The $\mathcal{M}_7$ can be obtained from the lowest order Born spin amplitude $\beta^0$ of $Z$ exchange by multiplication by two soft photon factors, for each of the bremsstrahlung photons, exactly as it should be in an exponentiation prescription. The factorization properties can be easily seen if a rather trivial manipulation on the Dirac algebra is performed.

In the case of diagrams with $W$ exchange, the choice of the extrapolation procedure is more complex, because of the dependence on the photon momenta of the transfers in the $W$ propagators. That is also the reason why triple and quartic gauge couplings appear. If the kinematical configurations of more than two explicit hard photons are taken, then the transfers calculated for the $W$ propagators can be defined in several ways. Our choice, used at present in KKMC [7], is inspired by leading-log considerations. For lowest order ($\beta^0$) and if there are no additional photons, the transfer, $t_0$, can be calculated either as
(i) $t_0 = (p_c - p_a)^2$ or
(ii) $t_0 = (p_d - p_b)^2$.

If there is a photon collinear to $p_b$ the first choice is closer to the transfer dominating the higher-order (i.e. single bremsstrahlung) spin amplitude. In general the choice (i) is thus more favored if the total four-momentum carried out by the sum of all photons is pointing rather into the direction of $p_b$ than $p_a$. Otherwise the second choice is better. In the case of single (or double) photon emission the choice of how the transfers are calculated is basically the same. The only difference is that the photons explicitly included in the particular contribution to $\beta^1$ or $\beta^2$ should not contribute to the sum of the photons mentioned above. The choice of the pair ($p_a$, $p_c$ or $p_b$, $p_d$) used in the calculation for the transfers must be taken also in the calculation of the algebraic expressions originating from a direct $W$ interaction with photons for gauge invariance to hold.

## 5 Summary

The purpose of the present paper was first of all practical: to close the gap in the documentation of the Monte Carlo program KKMC based on exponentiation [7] and used in the interpretation of the LEP data for neutrino pair production. We started with the presentation of well-known, tree level spin amplitudes for the $e^+e^- \to \nu_e\bar{\nu}_e\gamma$ and $e^+e^- \to \nu_e\bar{\nu}_e\gamma\gamma$ processes. To organize these terms we used an expansion with respect to the contact interaction for the $W$ exchange. We have shown how the properties of gauge invariance separate the results of the spin-amplitude calculations for a process with a $t$-channel contribution into amplitudes of $s$-channel exchange, and the rest, which is free of most of the singularities. We also were able to identify further gauge invariant parts of the amplitudes and order them accordingly to the level of a singularity. In particular, the parts proportional to the inverse of the photon energies (i.e. corresponding to an infrared singularity), the remaining parts proportional to the inverse of the product of fermion and photon direction vectors (i.e. of the type of collinear singularity), as well as the residual finite parts could be identified in a rather natural way. By comparison with amplitudes for the diagrams involving $s$-channel $Z$ exchange we were able to include the singular parts into the exponentiated factors and the rest could be treated up to the fixed order[12]. We could observe a certain pattern of universality, which may be of broader interest. General principles defining semi-automatic rules for gauge separation are listed in the first and last paragraphs of Sect. 3; see also footnote 10. The solution of spin amplitudes used for the neutrino mode of KKMC can be understood as a prototype, where a fully exclusive calculation for the production of two fermion final states through the $s$-channel only is the basis for a more complex one and the process where the $t$-channel exchanges contribute as well.

The question of how general the separation methods presented here are obviously is of interest. Separation into parts proportional to the different powers of the photon momenta is at the heart of the exponentiation scheme and is obviously not new. On the other hand, separation of spin amplitudes into finer parts seems to be a novel result. Similar separation properties, for the final-state radiation in decays of resonances, were helpful to develop the PHOTOS generator [16,17] into a multiple photon version [15] and into a version with a better emission from the $W$ decays [13]. However, in that case the additional complexity has to be addressed. It originates from the necessity to relate the four-momenta of the outgoing fermions for processes with a distinct number of photons, a difficulty obviously absent for initial-state radiation. On the other hand, in decays, where the precision of the PHOTOS algorithm was studied ($W$'s and $Z$'s) the internal complexity of *fixed-order* amplitudes is significantly smaller than in electron neutrino pair production. The recent paper of [18] is organized mainly around the numerical results of the PHOTOS tests. A more rigorous discussion of the principles of the multiple photon

[12] Let us stress that some parts of the results presented here are expected from the properties of U(1) gauge symmetry and the corresponding Ward identities. They have been known already for a long time, and similar ones are known in the context of QCD as well, but so far were discussed in the context either of inclusive or semi-inclusive quantities or, if for fully differential distributions of multiple particle final states, then with simplifications.

algorithm of PHOTOS is still missing. However, because the PHOTOS algorithm is of the parton shower type, the use of the properties of the spin amplitudes in its validation provide a hint that some of the methods presented here may find a way into a broader class of applications. They point toward a possible hierarchical organization of spin amplitudes with consecutive levels of simplifications, which can be restored back. At present, this highly speculative point cannot be supported by any theoretically solid argument, but only by examples. However, from the inspection of the presented QED calculation, the conjecture that the separation properties are more general than for the emissions from incoming and outgoing fermions and intermediate $W$'s seems to be natural (and it is also useful). Finally, let us point out similar observation as ours [19]: the case of the virtual, single loop corrections for the $e^+e^- \to \nu_e\bar{\nu}_e$ and $e^+e^- \to \nu_e\bar{\nu}_e H$ processes.

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